ON TRANSVERSAL INFINITESIMAL AUTOMORPHISMS FOR HARMONIC FOLIATIONS

ABSTRACT. In this paper we consider a harmonic Riemannian foliation \( \mathcal{F} \), and study the transversal infinitesimal automorphisms of \( \mathcal{F} \) with certain additional properties like being transversal conformal or Killing (= metric). Such automorphisms (modulo Killing automorphisms) are related to the stability of \( \mathcal{F} \). A special study is made for the case of a foliation with constant transversal scalar curvature, and more particularly with transversal Ricci curvature proportional to the transversal metric (Einstein foliation).

1. INTRODUCTION

Let \( \mathcal{F} \) be a transversally oriented foliation on a compact oriented Riemannian manifold \((M, g_M)\). It is given by an exact sequence of vector bundles

\[
0 \to L \to TM \xrightarrow{\pi} Q \to 0,
\]

where \( L \) is the tangent bundle and \( Q \) the normal bundle of \( \mathcal{F} \). We have an associated exact sequence of Lie algebras

\[
0 \to \Gamma L \to V(\mathcal{F}) \xrightarrow{\pi} \Gamma Q^\perp \to 0,
\]

where \( V(\mathcal{F}) \) denotes the algebra of infinitesimal automorphisms of \( \mathcal{F} \), and \( \Gamma Q^\perp \) the portion of \( \Gamma Q \) invariant under the action of \( L \) by Lie derivatives \(([5], [12])\). The foliation is assumed to be Riemannian with bundle-like metric \( g_M \), and holonomy invariant induced metric \( g_Q \) on \( Q \cong L^\perp \). The unique metric and torsion-free connection in \( Q \) is denoted by \( \nabla \) \(([4], [12])\). Associated to \( \nabla \) are transversal curvature data, in particular, the (transversal) Ricci operator \( \rho_V: Q \to Q \) and the (transversal) scalar curvature \( c_V = \text{trace } \rho_V \) \(([5])\). In this paper we study geometric properties of infinitesimal automorphisms \( Y \in V(\mathcal{F}) \). For \( Y \in V(\mathcal{F}) \) the transversal part \( \pi(Y) \) of \( Y \) is also denoted by \( \hat{Y} \) and \( \omega \) will stand for the basic 1-form associated to \( \hat{Y} \) by \((g_Q)\) duality.

Recall that the basic forms are given by

\[
\Omega^*_B = \{ \omega \in \Omega^*(M) \mid i(X)\omega = 0, \Theta(X)\omega = 0 \text{ for all } X \in \Gamma L \}.
\]

The exterior differential \( d \) restricts to \( d_B: \Omega^*_B \to \Omega^{*+1}_B \). The adjoint of \( d_B \), with

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respect to the induced scalar product $\langle , \rangle_B$ on $\Omega_B$, is denoted by $\delta_B: \Omega^*_B \to \Omega^*_{B^{-1}}$ and we then have the (basic) Laplacian $\Delta_B = \delta_B d_B + d_B \delta_B$. The relation expressing $\delta_B$ in terms of $\nabla$ involves the mean curvature form of $\mathcal{F}$ which, in this paper, we assume to be zero. In other words, $\mathcal{F}$ is assumed to be harmonic, i.e., all leaves of $\mathcal{F}$ are minimal ([4]).

By [8], [9], the De Rham–Hodge decomposition generalizes to a decomposition

$$\Omega^*_B \cong \operatorname{im} d_B \oplus \operatorname{im} \delta_B \oplus \mathcal{H}^*_B,$$

into mutually orthogonal subspaces, with finite dimensional space of harmonic basic forms $\mathcal{H}^*_B = \ker \Delta_B$.

In [10] the operators $\delta^*, \delta$ occurring in the Berger–Ebin decomposition [2] were generalized to the foliation context.

$$\delta^*: \Gamma Q^* \to \Gamma S^2 Q^*, S^2 = \text{symmetric square},$$

is given by

$$(\delta^* \omega)(V, W) = \frac{1}{2} \{ (\nabla_V \omega)(W) + \nabla_W \omega)(V) \}, \quad \omega \in \Gamma Q^*, \quad V, W \in \Gamma Q.$$

It maps the basic 1-forms $\Omega_B^* \subset \Gamma Q^*$ to basic symmetric 2-forms, i.e., those killed by $i(X), \Theta(X)$ for all $X \in \Gamma L$. For the present purpose it suffices to know that $\delta: \Gamma S^2 Q^* \to \Gamma Q^*$ defined in [10] restricts on basic forms to the adjoint of $\delta^*$. The fundamental identities for $Y \in V(\mathcal{F})$ and $\omega = g_Q$-dual of $\nabla$, in the case of a harmonic Riemannian foliation $\mathcal{F}$, are then ([10])

$$2 \delta \delta^* \omega = - \text{trace} \nabla^2 \omega - \rho_V(\omega) + d_B \delta B \omega,$$

$$\text{div}_B Y = - \delta^* \omega = (\delta^* \omega, g_Q),$$

$$|\delta^* \omega - \frac{1}{q} \text{div}_B Y \cdot g_Q|^2 = |\delta^* \omega|^2 - \frac{1}{q} (\text{div}_B Y)^2, \quad q = \text{codim} \mathcal{F}. $$

Let $Y \in V(\mathcal{F})$. Then $\nabla$ is divergence free if $\text{div}_B Y = 0$, a transversal Jacobi automorphism if $Y \in \ker J_V$, where $J_V = - \text{trace} \nabla^2 - \rho_V$ is the Jacobi operator, a transversal Killing automorphism if $\Theta(Y) g_Q = 0$ and transversal conformal if $\Theta(Y) g_Q = \mu \cdot g_Q$ for some basic function $\mu$. These properties can be equivalently expressed in terms of the $g_Q$-dual $\omega$ by $\delta_B \omega = 0$, trace $\nabla^2 \omega + \rho_V(\omega) = 0$, $\delta^* \omega = 0$ and $\delta^* \omega = - (1/q) \delta_B \omega \cdot g_Q$, respectively ([10]). This motivates the introduction of the following concept.

2. $\sigma$-Automorphisms

Given $Y \in V(\mathcal{F})$, $\nabla$ is said to be a $\sigma$-automorphism for $\sigma \in \mathbb{R}$ if (the $g_Q$-dual)$\omega$ satisfies

$$- \text{trace} \nabla^2 \omega - \rho_V(\omega) + \sigma d_B \delta_B \omega = 0.$$

or equivalently,
\begin{equation}
\Delta \omega - 2\rho \nabla(\omega) + \sigma d_B \delta_B \omega = 0,
\end{equation}
where we use the Bochner–Weitzenböck formula \( \Delta \omega = -\text{trace} \nabla^2 \omega + \rho \nabla(\omega) \).

2.3. EXAMPLE. By (1.1) and (1.2), for \( Y \in V(\mathcal{F}) \), \( Y \) is transversal Killing iff \( Y \) is a divergence-free Jacobi automorphism. Hence, a transversal Killing \( Y \) is a \( \sigma \)-automorphism for all \( \sigma \in \mathbb{R} \). Moreover, a \( \sigma \)-automorphism is transversal Killing iff it is divergence free.

2.4. EXAMPLE. The transversal Jacobi automorphisms are precisely the 0-automorphisms of \( \mathcal{F} \).

2.5. EXAMPLE. The transversal conformal automorphisms are the \((1 - 2/q)\)-automorphisms of \( \mathcal{F} \). Indeed, \( \tilde{Y} \) is conformal iff \( \delta^* \omega = -(1/q)(d_B \delta_B \omega, g) \). Applying \( \delta \) we get \( \delta \delta^* \omega = (1/q)d_B \delta_B \omega \). Conversely, this identity implies conformality since, by (1.3),
\begin{align*}
||\delta^* \omega - \frac{1}{q} \text{div}_B \tilde{Y} \cdot g\omega||^2 &= ||\delta^* \omega||^2 - \frac{1}{q} ||\delta_B \omega||^2 \\
&= \langle \delta \delta^* \omega, \omega \rangle - \frac{1}{q} \langle d_B \delta_B \omega, \omega \rangle = 0
\end{align*}
and conformality follows. Now the claim is a direct consequence of (1.1).

2.6. EXAMPLE. The transversal projective automorphisms are precisely the \((-2/(q+1))\)-automorphisms of \( \mathcal{F} \).

For \( \sigma \in \mathbb{R} \), we introduce the vectorspace
\[ A_\sigma = \begin{cases}
\{ \sigma \text{-automorphisms} \} \\
\{ \text{transversal Killing automorphisms} \}
\end{cases} \]
and define
\[ \Sigma = \{ \sigma \in \mathbb{R} | A_\sigma \neq \{0\} \} \).

2.7. THEOREM. Let \( \mathcal{F} \) be a transversally oriented harmonic Riemannian foliation on a compact oriented Riemannian manifold \( M \). If \( \mathcal{F} \) is stable, then \( \Sigma \leq 0 \).

Proof. A harmonic foliation is a critical point of the energy functional \( E(\mathcal{F}) = \frac{1}{2} \int_M |\pi|^2 \cdot \text{vol}(M)([4]) \). The second variation for \( E \) for a special variation \( \mathcal{F}_t \) of \( \mathcal{F}_0 = \mathcal{F} \) given by \( X \in \Gamma Q \) is, according to [5], given by
\[ \frac{\partial^2}{\partial t^2} E(\mathcal{F}_t)|_{t=0} = \langle J_v X, X \rangle. \]
\( \mathcal{F} \) is stable if \( \langle J_v X, X \rangle \geq 0 \) for all \( X \in \Gamma Q \). Now assume that \( \tilde{Y} \) is a \( \sigma \)-
automorphism for $\sigma > 0$. Then we have
\[
\langle J_\sigma \tilde{Y}, \tilde{Y} \rangle = -\sigma \langle d_B \delta_B \omega, \omega \rangle = -\sigma \| \delta_B \omega \|^2 < 0,
\]
where $\omega = g_{q^*-}\text{dual of } \tilde{Y}$. It follows that $\tilde{Y}$ is divergence free and hence Killing, i.e. $\sigma \notin \Sigma$.

2.8. EXAMPLE. The instability of harmonic foliations on $\mathbb{S}^n(n > 2)$ with $q > 2$ was proved in [6].

2.9. EXAMPLE. If $\mathcal{F}$ has a dense leaf in $M$ then $\Sigma = \emptyset$. Indeed, if $\tilde{Y}$ is a $\sigma$-automorphism then the function $\text{div}_B \tilde{Y}$ is basic and hence constant on $M$. By the transversal divergence theorem ([10]) $\int_M \text{div}_B \tilde{Y} \cdot \text{vol}(M) = 0$ and $\tilde{Y}$ is divergence free, hence Killing.

2.10. THEOREM. Let $\mathcal{F}$ and $M$ be as in Theorem 2.7 and assume that the transversal scalar curvature $c_v$ is constant:

(i) if $c_v > 0$, then $\Sigma > -1$;
(ii) if $c_v \leq 0$, then $\Sigma \leq -1$ (similarly, with sharp inequalities).

Proof. We show (i), the proof of (ii) being analogous. Let $c_v > 0$ and $\sigma \in \Sigma$ and choose a $\sigma$-automorphism $\tilde{Y}$ with $\text{div}_B \tilde{Y} \neq 0$. Equation (2.2) can be rewritten as
\[
(1 + \sigma)\Delta_B \omega - 2\rho_v(\omega) - \sigma \delta_B d_B \omega = 0.
\]
Applying $\delta_B$ we have
\[
(1 + \sigma)\Delta_B \delta_B \omega = 2\delta_B \rho_v(\omega) = \frac{2}{q}c_v \cdot \delta_B \omega,
\]
where the last equality is obtained by direct computation using $c_v = \text{const}$. Taking the global scalar product with $\delta_B \omega$ we get
\[
(1 + \sigma) \langle \Delta_B \delta_B \omega, \delta_B \omega \rangle = \frac{2}{q}c_v \| \delta_B \omega \|^2 > 0.
\]
Note that $\| \delta_B \omega \|^2$ does not vanish since otherwise $\tilde{Y}$ would be divergence free.

Similarly, $\langle \Delta_B \delta_B \omega, \delta_B \omega \rangle = \|\delta_B d_B \delta_B \omega, \delta_B \omega \| = \| d_B \delta_B \omega \|^2 > 0$ since otherwise $\langle d_B \delta_B \omega, \omega \rangle = \| \delta_B \omega \|^2 = 0$ would follow. Thus $\sigma > -1$, which completes the proof.

3. TRANSVERSALLY EINSTEIN FOLIATIONS

$\mathcal{F}$ is said to be transversally Einstein if $\rho_v = (\lambda/2) i d_B : Q \to Q$ for some $\lambda \in \mathbb{R}$. In particular, $c_v = (\lambda/2) q = \text{const.}$ and, by the previous theorem, for $\lambda \leq 0$, all transversal Jacobi, conformal and projective automorphisms are Killing, provided that $q \geq 2$. 
3.1. THEOREM. Let the harmonic foliation $\mathcal{F}$ be transversally Einstein with $\lambda > 0$. Then $\Sigma \subset [-1, 1 - (2/q)]$. If $\Sigma$ is infinite then it forms a decreasing sequence converging to $-1$. Moreover, for a $\sigma$-automorphism $\bar{Y}$, we have the decomposition

$$\bar{Y} = \bar{Y}_1 + \text{grad}_n \mu,$$

where $\bar{Y}_1$ is transversal Killing and $\mu$ is a basic scalar which can be uniquely characterized as an eigenfunction of $\Lambda^M$ on $\mathcal{M}$ with eigenvalue $\lambda/(1 + \sigma)$.

Proof. $\Sigma > -1$ by (i) of the previous theorem. We now show that the decomposition (3.2) holds for an arbitrary $\sigma$-automorphism $\bar{Y}$, $\sigma \in \mathbb{R}$. In the present situation, (2.2) takes the form

$$\Delta_{\beta} \omega - \lambda \omega + \sigma \text{ d}_{\beta} \delta_{\beta} \omega = 0.$$

Consider the De Rham–Hodge decomposition of $\omega$ ([9])

$$\omega = \text{d}_{\beta} \mu + \delta_{\beta} \beta + \pi_{\beta} \omega,$$

where $\mu \in \Omega^0_{\beta}$, $\beta \in \Omega^2_{\beta}$, and $\pi_{\beta} : \Omega^1_{\beta} \to \mathcal{H}_{\beta}^1$ denotes the projection onto the subspace of harmonic 1-forms. Substituting this into (3.3) and using orthogonality we get

$$\text{d}_{\beta}((1 + \sigma)\Delta_{\beta} \mu - \lambda \mu) = 0,$$

$$\delta_{\beta}(\Delta_{\beta} \beta - \lambda \beta) = \Delta_{\beta} \delta_{\beta} \beta - \lambda \delta_{\beta} \beta = 0,$$

$$\pi_{\beta} \omega = 0.$$

The first equation says that $(1 + \sigma)\Delta_{\beta} \mu - \lambda \mu$ is constant and so, modifying $\mu$ with an additive constant, we obtain

$$\Delta_{\beta} \mu = \lambda \mu.$$

As $\Lambda^M = \Delta_{\beta}$ on basic scalars, it shows that $\mu$ is a uniquely determined basic scalar which is an eigenfunction of $\Lambda^M$ with eigenvalue $\lambda/(1 + \sigma)$. As $\text{d}_{\beta} \mu$ is the dual of $\text{grad}_n \mu$ it remains to show that the dual $\bar{Y}_1$ of $\delta_{\beta} \beta$ is Killing. Now, the second equation says that $\bar{Y}_1$ is Jacobi. On the other hand, $\text{div}_n \bar{Y}_1 = -\delta_{\beta} \beta = 0$ and so $\bar{Y}_1$ is Killing. To complete the proof of the theorem it remains to show that $\Sigma \leq 1 - (1/q)$. So, let $\sigma \in \Sigma$ and $\bar{Y}$ a $\sigma$-automorphism with dual $\omega \in \Omega^1_{\beta}$. By (3.2) we may assume that $\omega = \text{d}_{\beta} \mu$, where $\mu (\neq 0)$ is a basic scalar with

$$\Delta^M \mu = \frac{\lambda}{1 + \sigma} \mu.$$
By the Bochner–Weitzenböck formula

$$-\frac{1}{2} \Delta_B (|\omega|^2) = |\nabla \omega|^2 - (\Delta_B \omega, \omega) + (d_B \omega, \omega) + \text{div}_B Z,$$

where $Z \in V(\mathcal{F})$ is defined by $\omega$. In the present situation, this reduces to

$$-\frac{1}{2} \Delta_B (|d_B \mu|^2) = |\nabla d_B \mu|^2 + \lambda \left( \frac{1}{2} - \frac{1}{1 + \sigma} \right) |d_B \mu|^2 + \text{div}_B Z.$$

Integrating and using the transversal divergence theorem ([10]), we get

$$0 = |\nabla d_B \mu|^2 + \frac{1}{2} |d_B \mu|^2,$$

or equivalently,

$$0 = |\nabla d_B \mu|^2 + \left( \frac{\sigma + 1}{2} - 1 \right) |d_B \mu|^2,$$

where we used $|d_B \mu|^2 = \langle \Delta_B \mu, \mu \rangle = ((\sigma + 1)/\lambda) |\Delta_B \mu|^2$. Now, by the Cauchy–Schwarz inequality (on the model space of $\mathcal{F}$)

$$|\Delta d_B \mu|^2 \geq \frac{1}{q} |\Delta_B \mu|^2,$$

and we obtain

$$0 \geq \left( \frac{1}{q} + \frac{\sigma + 1}{2} - 1 \right) |\Delta_B \mu|^2.$$

As $|\Delta_B \mu|^2 > 0$, the inequality $\sigma \leq 1 - (2/q)$ follows.

Let $\text{Spec}_B \subset \text{Spec}(M)$ denote the set of positive eigenvalues of $\Delta^M$ for which there exist basic eigenfunctions. Then $\text{Spec}_B = \{ \lambda_k \}_{k=1}^K$ with $0 \leq K \leq \infty$, which, if $K = \infty$, diverges to $\infty$. By the previous theorem, $\Sigma = \{ \lambda_k = (\lambda_k - 1) \}_{k=1}^K$. Moreover, $A_{\sigma_k} \cong$ vector space of basic eigenfunctions of $\Delta^M$ with eigenvalue $\lambda_k$. Note also that if $\mathcal{F}$ is given by the fibres of a harmonic Riemannian submersion $f: M \to N$ then $\Omega^B_\sigma = \Omega^\sigma(N)$ and $\text{Spec}_B = \text{Spec}(N) \setminus \{ 0 \}$, $K = \infty$ and $\dim A_{\sigma_k} =$ multiplicity of $\lambda_k$ as an eigenvalue for $\Delta^N$.

3.4. EXAMPLE. If $\mathcal{F}$ is defined by the fibres of the Hopf map $f: S^{2n-1} \to \mathbb{C} P^n$ then $q = 2n$, $\rho_\mathcal{F} = \rho^{\mathbb{C} P^n} = 2(n+1)\mathrm{id}$, that is,

$$\lambda = 4(n+1), \text{Spec}_B = \{ \lambda_k = 4k(n+k) \}_{k=1}^{\infty},$$

and

$$\sum \left\{ \sigma_k = \frac{n+1}{k(k+n)} - 1 \right\}_{k=1}^{\infty}.$$
As $\Sigma \leq 0$, for $n \geq 2$, every transversal conformal automorphism is Killing. On the other hand, for $n = 1$, i.e. $q = 2$, we have $1 - (2/q) = \sigma_1 = 0$ and $\dim A_0 = 3$.

3.5. EXAMPLE. Let $\mathcal{F}$ be defined by the fibres of a harmonic Riemannian submersion $f: M \to S^n, n \geq 2$. (A large class of such maps are constructed in [1].) Then $q = n, \rho_S = \rho S^n = (n - 1) \text{id}$; that is,

$$\hat{\mathcal{F}} = 2(n - 1), \text{Spec}_B = \{\hat{\mathcal{F}}_k = k(k + n - 1)\}_{k=1}^{\infty}$$

and

$$\sum = \left\{\sigma_k = \frac{2(n - 1)}{k(k + n - 1)} - 1\right\}_{k=1}^{\infty}.$$

(As $\sigma_1 = 1 - (2/q)$ the bounds for $\Sigma$ in Theorem 3.1 are the best possible.)

For $n \neq 2, 0 \notin \Sigma$ and so every transversal Jacobi automorphism is Killing. For $n = 2, \sigma_1 = 0$ and again $\dim A_0 = 3$.

REFERENCES

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