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# ON TRANSVERSAL INFINITESIMAL AUTOMORPHISMS FOR HARMONIC FOLIATIONS

ABSTRACT. In this paper we consider a harmonic Riemannian foliation $\mathscr{F}$, and study the transversal infinitesimal automorphisms of $\mathscr{F}$ with certain additional properties like being transversal conformal or Killing (= metric). Such automorphisms (modulo Killing automorphisms) are related to the stability of $\mathscr{F}$. A special study is made for the case of a foliation with constant transversal scalar curvature, and more particularly with transversal Ricci curvature proportional to the transversal metric (Einstein foliation).

## 1. INTRODUCTION

Let $\mathscr{F}$ be a transversally oriented foliation on a compact oriented Riemannian manifold $(M, g_M)$. It is given by an exact sequence of vector bundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0,$$

where $L$ is the tangent bundle and $Q$ the normal bundle of $\mathscr{F}$. We have an associated exact sequence of Lie algebras

$$0 \to \Gamma L \to V(\mathscr{F}) \xrightarrow{\pi} \Gamma Q^L \to 0,$$

where $V(\mathscr{F})$ denotes the algebra of infinitesimal automorphisms of $\mathscr{F}$, and $\Gamma Q^L$ the portion of $\Gamma Q$ invariant under the action of $L$ by Lie derivatives ([5], [12]). The foliation is assumed to be Riemannian with bundle-like metric $g_M$, and holonomy invariant induced metric $g_Q$ on $Q \cong L^{\perp}$. The unique metric and torsion-free connection in $Q$ is denoted by $\nabla$ ([4], [12]). Associated to $\nabla$ are transversal curvature data, in particular, the (transversal) Ricci operator $\rho_\nabla: Q \to Q$ and the (transversal) scalar curvature $c_\nabla =$ trace $\rho_\nabla$ ([5]). In this paper we study geometric properties of infinitesimal automorphisms $Y \in V(\mathscr{F})$. For $Y \in V(\mathscr{F})$ the transversal part $\pi(Y)$ of $Y$ is also denoted by $\bar{Y}$ and $\omega$ will stand for the basic 1-form associated to $\bar{Y}$ by $(g_Q)$ duality.

Recall that the basic forms are given by

$$\Omega_B^\bullet = \{\omega \in \Omega^\bullet(M) \mid i(X)\omega = 0,\ \Theta(X)\omega = 0 \quad \text{for all } X \in \Gamma L\}.$$

The exterior differential d restricts to $d_B: \Omega_B^\bullet \to \Omega_B^{\bullet+1}$. The adjoint of $d_B$, with

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respect to the induced scalar product $\langle\ ,\ \rangle_B$ on $\Omega_B$, is denoted by $\delta_B\colon \Omega_B^\bullet \to \Omega_B^{\bullet-1}$ and we then have the (basic) Laplacian $\Delta_B = \delta_B d_B + d_B\delta_B$. The relation expressing $\delta_B$ in terms of $\nabla$ involves the mean curvature form of $\mathscr{F}$ which, in this paper, we assume to be zero. In other words, $\mathscr{F}$ is assumed to be harmonic, i.e. all leaves of $\mathscr{F}$ are minimal ([4]).

By [8], [9], the De Rham–Hodge decomposition generalizes to a decomposition

$$\Omega_B^\bullet \cong \operatorname{im} d_B \oplus \operatorname{im} \delta_B \oplus \mathscr{H}_B^\bullet,$$

into mutually orthogonal subspaces, with finite dimensional space of harmonic basic forms $\mathscr{H}_B^\bullet = \ker \Delta_B$.

In [10] the operators $\delta^*, \delta$ occurring in the Berger–Ebin decomposition [2] were generalized to the foliation context.

$$\delta^*\colon \Gamma Q^* \to \Gamma S^2 Q^*,\ S^2 = \text{symmetric square},$$

is given by

$$(\delta^*\omega)(V, W) = \tfrac{1}{2}\{(\nabla_V\omega)(W) + \nabla_W\omega)(V)\},\quad \omega \in \Gamma Q^*,\ V, W \in \Gamma Q.$$

It maps the basic 1-forms $\Omega_B^1 \subset \Gamma Q^*$ to basic symmetric 2-forms, i.e. those killed by $i(X)$, $\Theta(X)$ for all $X \in \Gamma L$. For the present purpose it suffices to know that $\delta\colon \Gamma S^2 Q^* \to \Gamma Q^*$ defined in [10] restricts on basic forms to the adjoint of $\delta^*$. The fundamental identities for $Y \in V(\mathscr{F})$ and $\omega = g_Q$-dual of $\bar{Y}$, in the case of a harmonic Riemannian foliation $\mathscr{F}$, are then ([10])

$$(1.1)\qquad 2\delta\delta^*\omega = -\operatorname{trace}\nabla^2\omega - \rho_\nabla(\omega) + d_B\delta_B\omega,$$

$$(1.2)\qquad \operatorname{div}_B\bar{Y} = -\delta_B\omega = (\delta^*\omega, g_Q),$$

$$(1.3)\qquad |\delta^*\omega - \tfrac{1}{q}\operatorname{div}_B\bar{Y}\cdot g_Q|^2 = |\delta^*\omega|^2 - \tfrac{1}{q}(\operatorname{div}_B\bar{Y})^2,\quad q = \operatorname{codim}\mathscr{F}.$$

Let $Y \in V(\mathscr{F})$. Then $\bar{Y}$ is divergence free if $\operatorname{div}_B\bar{Y} = 0$, a transversal Jacobi automorphism if $Y \in \ker J_\nabla$, where $J_\nabla = -\operatorname{trace}\nabla^2 - \rho_\nabla$ is the Jacobi operator, a transversal Killing automorphism if $\Theta(Y)g_Q = 0$ and transversal conformal if $\Theta(Y)g_Q = \mu\cdot g_Q$ for some basic function $\mu$. These properties can be equivalently expressed in terms of the $g_Q$-dual $\omega$ by $\delta_B\omega = 0$, trace $\nabla^2\omega + \rho_\nabla(\omega) = 0$, $\delta^*\omega = 0$ and $\delta^*\omega = -(1/q)\delta_B\omega\cdot g_Q$, respectively ([10]). This motivates the introduction of the following concept.

## 2. $\sigma$-Automorphisms

Given $Y \in V(\mathscr{F})$, $\bar{Y}$ is said to be a $\sigma$-automorphism for $\sigma \in \mathbb{R}$ if (the $g_Q$-dual)$\omega$ satisfies

$$(2.1)\qquad -\operatorname{trace}\nabla^2\omega - \rho_\nabla(\omega) + \sigma\, d_B\delta_B\omega = 0,$$

or equivalently,

$$(2.2) \qquad \Delta_B\omega - 2\rho_\nabla(\omega) + \sigma\, d_B\delta_B\omega = 0,$$

where we use the Bochner–Weitzenböck formula $\Delta_B\omega = -\operatorname{trace} \nabla^2\omega + \rho_\nabla(\omega)$.

2.3. EXAMPLE. By (1.1) and (1.2), for $Y \in V(\mathscr{F})$. $\bar{Y}$ is transversal Killing iff $\bar{Y}$ is a divergence-free Jacobi automorphism. Hence, a transversal Killing $\bar{Y}$ is a $\sigma$-automorphism for all $\sigma \in \mathbb{R}$. Moreover, a $\sigma$-automorphism is transversal Killing iff it is divergence free.

2.4. EXAMPLE. The transversal Jacobi automorphisms are precisely the 0-automorphisms of $\mathscr{F}$.

2.5. EXAMPLE. The transversal conformal automorphisms are the $(1 - 2/q)$-automorphisms of $\mathscr{F}$. Indeed, $\bar{Y}$ is conformal iff $\delta^*\omega = -(1/q)\delta_B\omega \cdot g_Q$. Applying $\delta$ we get $\delta\delta^*\omega = (1/q)\, d_B\delta_B\omega$. Conversely, this identity implies conformality since, by (1.3),

$$\begin{aligned} &\|\delta^*\omega - \tfrac{1}{q}\operatorname{div}_B\bar{Y}\cdot g_Q\|^2 = \|\delta^*\omega\|^2 - \tfrac{1}{q}\|\delta_B\omega\|^2 = \\ &= \langle\delta\delta^*\omega, \omega\rangle - \tfrac{1}{q}\langle d_B\delta_B\omega, \omega\rangle = 0 \end{aligned}$$

and conformality follows. Now the claim is a direct consequence of (1.1).

2.6. EXAMPLE. The transversal projective automorphisms are precisely the $(-2/(q+1))$-automorphisms of $\mathscr{F}$.

For $\sigma \in \mathbb{R}$, we introduce the vectorspace

$$A_\sigma = \frac{\{\sigma\text{-automorphisms}\}}{\{\text{transversal Killing automorphisms}\}}$$

and define

$$\Sigma = \{\sigma \in \mathbb{R} \mid A_\sigma \neq \{0\}\}.$$

2.7. THEOREM. *Let $\mathscr{F}$ be a transversally oriented harmonic Riemannian foliation on a compact oriented Riemannian manifold M. If $\mathscr{F}$ is stable, then* $\Sigma \leqslant 0$.

*Proof.* A harmonic foliation is a critical point of the energy functional $E(\mathscr{F}) = \frac{1}{2}\int_M |\pi|^2 \cdot \operatorname{vol}(M)$([4]). The second variation for $E$ for a special variation $\mathscr{F}_t$ of $\mathscr{F}_0 = \mathscr{F}$ given by $X \in \Gamma Q$ is, according to [5], given by

$$\frac{\partial^2}{\partial t^2} E(\mathscr{F}_t)_{t=0} = \langle J_\nabla X, X\rangle.$$

$\mathscr{F}$ is stable if $\langle J_\nabla X, X\rangle \geqslant 0$ for all $X \in \Gamma Q$. Now assume that $\bar{Y}$ is a $\sigma$-

automorphism for $\sigma > 0$. Then we have

$$\langle J_\nabla \bar{Y}, \bar{Y} \rangle = -\sigma \langle d_B \delta_B \omega, \omega \rangle = -\sigma \|\delta_B \omega\|^2 \leqslant 0,$$

where $\omega = g_Q$-dual of $\bar{Y}$. It follows that $\bar{Y}$ is divergence free and hence Killing, i.e. $\sigma \notin \Sigma$.

2.8. EXAMPLE. The instability of harmonic foliations on $S^n (n > 2)$ with $q > 2$ was proved in [6].

2.9. EXAMPLE. If $\mathscr{F}$ has a dense leaf in $M$ then $\Sigma = \emptyset$. Indeed, if $\bar{Y}$ is a $\sigma$-automorphism then the function $\operatorname{div}_B \bar{Y}$ is basic and hence constant on $M$. By the transversal divergence theorem ([10]) $\int_M \operatorname{div}_B \bar{Y} \cdot \operatorname{vol}(M) = 0$ and $\bar{Y}$ is divergence free, hence Killing.

2.10. THEOREM. *Let $\mathscr{F}$ and $M$ be as in Theorem 2.7 and assume that the transversal scalar curvature $c_\nabla$ is constant:*

(i) *if $c_\nabla > 0$, then $\Sigma > -1$;*
(ii) *if $c_\nabla \leqslant 0$, then $\Sigma \leqslant -1$ (similarly, with sharp inequalities).*

*Proof.* We show (i), the proof of (ii) being analogous. Let $c_\nabla > 0$ and $\sigma \in \Sigma$ and choose a $\sigma$-automorphism $\bar{Y}$ with $\operatorname{div}_B \bar{Y} \not\equiv 0$. Equation (2.2) can be rewritten as

$$(1 + \sigma)\Delta_B \omega - 2\rho_\nabla(\omega) - \sigma\, \delta_B d_B \omega = 0.$$

Applying $\delta_B$ we have

$$(1 + \sigma)\Delta_B \delta_B \omega = 2\delta_B \rho_\nabla(\omega) = \tfrac{2}{q} c_\nabla \cdot \delta_B \omega,$$

where the last equality is obtained by direct computation using $c_\nabla = \text{const}$. Taking the global scalar product with $\delta_B \omega$ we get

$$(1 + \sigma)\langle \Delta_B \delta_B \omega, \delta_B \omega \rangle = \tfrac{2}{q} c_B \|\delta_B \omega\|^2 > 0.$$

Note that $\|\delta_B \omega\|^2$ does not vanish since otherwise $\bar{Y}$ would be divergence free.

Similarly, $\langle \Delta_B \delta_B \omega, \delta_B \omega \rangle = \langle \delta_B d_B \delta_B \omega, \delta_B \omega \rangle = \|d_B \delta_B \omega\|^2 > 0$ since otherwise $\langle d_B \delta_B \omega, \omega \rangle = \|\delta_B \omega\|^2 = 0$ would follow. Thus $\sigma > -1$, which completes the proof.

## 3. Transversally Einstein Foliations

$\mathscr{F}$ is said to be transversally Einstein if $\rho_\nabla = (\lambda/2)\mathrm{id}_Q : Q \to Q$ for some $\lambda \in \mathbb{R}$. In particular, $c_\nabla = (\lambda/2)q = \text{const.}$ and, by the previous theorem, for $\lambda \leqslant 0$, all transversal Jacobi, conformal and projective automorphisms are Killing, provided that $q \geqslant 2$.

3.1. THEOREM. *Let the harmonic foliation $\mathscr{F}$ be transversally Einstein with $\lambda > 0$. Then $\Sigma \subset [-1, 1-(2/q)]$. If $\Sigma$ is infinite then it forms a decreasing sequence converging to $-1$. Moreover, for a $\sigma$-automorphism $\bar{Y}$, we have the decomposition*

(3.2) $$\bar{Y} = \bar{Y}_1 + \operatorname{grad}_B \mu,$$

*where $\bar{Y}_1$ is transversal Killing and $\mu$ is a basic scalar which can be uniquely characterized as an eigenfunction of $\Delta^M$ on $M$ with eigenvalue $\lambda/(1+\sigma)$.*

*Proof.* $\Sigma > -1$ by (i) of the previous theorem. We now show that the decomposition (3.2) holds for an arbitrary $\sigma$-automorphism $\bar{Y}, \sigma \in \mathbb{R}$. In the present situation, (2.2) takes the form

(3.3) $$\Delta_B\omega - \lambda\omega + \sigma\, \mathrm{d}_B\delta_B\omega = 0.$$

Consider the De Rham–Hodge decomposition of $\omega$ ([9])

$$\omega = \mathrm{d}_B\mu + \delta_B\beta + \pi_B\omega,$$

where $\mu \in \Omega_B^0$, $\beta \in \Omega_B^2$, and $\pi_B\colon \Omega_B^1 \to \mathscr{H}_B^1$ denotes the projection onto the subspace of harmonic 1-forms. Substituting this into (3.3) and using orthogonality we get

$$\mathrm{d}_B((1+\sigma)\Delta_B\mu - \lambda\mu) = 0,$$
$$\delta_B(\Delta_B\beta - \lambda\beta) = \Delta_B\delta_B\beta - \lambda\delta_B\beta = 0,$$
$$\pi_B\omega = 0.$$

The first equation says that $(1+\sigma)\Delta_B\mu - \lambda\mu$ is constant and so, modifying $\mu$ with an additive constant, we obtain

$$\Delta_B\mu = \frac{\lambda}{1+\sigma}\mu.$$

As $\Delta^M = \Delta_B$ on basic scalars, it shows that $\mu$ is a uniquely determined basic scalar which is an eigenfunction of $\Delta^M$ with eigenvalue $\lambda/(1+\sigma)$. As $\mathrm{d}_B\mu$ is the dual of $\operatorname{grad}_B\mu$ it remains to show that the dual $\bar{Y}_1$ of $\delta_B\beta$ is Killing. Now, the second equation says that $\bar{Y}_1$ is Jacobi. On the other hand, $\operatorname{div}_B\bar{Y}_1 = -\delta_B^2\beta = 0$ and so $\bar{Y}_1$ is Killing. To complete the proof of the theorem it remains to show that $\Sigma \leqslant 1-(1/q)$. So, let $\sigma \in \Sigma$ and $\bar{Y}$ a $\sigma$-automorphism with dual $\omega \in \Omega_B^1$. By (3.2) we may assume that $\omega = \mathrm{d}_B\mu$, where $\mu$ ($\not\equiv 0$) is a basic scalar with

$$\Delta^M\mu = \frac{\lambda}{1+\sigma}\mu.$$

By the Bochner–Weitzenböck formula

$$-\tfrac{1}{2}\Delta_B(|\omega|^2) = |\nabla\omega|^2 - (\Delta_B\omega,\omega) + (\rho_\nabla\omega,\omega) + \operatorname{div}_B Z,$$

where $Z \in V(\mathscr{F})$ is defined by $\omega$. In the present situation, this reduces to

$$-\tfrac{1}{2}\Delta_B\,(|\mathrm{d}_B\mu|^2) = |\nabla\,\mathrm{d}_B\mu|^2 + \lambda\left(\frac{1}{2} - \frac{1}{1+\sigma}\right)|\mathrm{d}_B\mu|^2 + \operatorname{div}_B Z.$$

Integrating and using the transversal divergence theorem ([10]), we get

$$0 = \|\nabla\,\mathrm{d}_B\mu\|^2 + \lambda\left(\frac{1}{2} - \frac{1}{1+\sigma}\right)\|\mathrm{d}_B\mu\|^2,$$

or equivalently,

$$0 = \|\nabla\,\mathrm{d}_B\mu\|^2 + \left(\frac{\sigma+1}{2} - 1\right)\|\mathrm{d}_B\mu\|^2.$$

where we used $\|\mathrm{d}_B\mu\|^2 = \langle\Delta_B\mu,\mu\rangle = ((\sigma+1)/\lambda)\|\Delta_B\mu\|^2$. Now, by the Cauchy–Schwartz inequality (on the model space of $\mathscr{F}$)

$$|\Delta\,\mathrm{d}_B\mu|^2 \geqslant \tfrac{1}{q}|\Delta_B\mu|^2$$

and we obtain

$$0 \geqslant \left(\frac{1}{q} + \frac{\sigma+1}{2} - 1\right)\|\Delta_B\mu\|^2.$$

As $\|\Delta_B\mu\|^2 > 0$, the inequality $\sigma \leqslant 1 - (2/q)$ follows.

Let $\mathrm{Spec}_B \subset \mathrm{Spec}(M)$ denote the set of positive eigenvalues of $\Delta^M$ for which there exist basic eigenfunctions. Then $\mathrm{Spec}_B = \{\lambda_k\}_{k=1}^K$ with $0 \leqslant K \leqslant \infty$, which, if $K = \infty$, diverges to $\infty$. By the previous theorem, $\Sigma = \{\sigma_k = (\lambda/\lambda_k) - 1\}_{k=1}^K$. Moreover, $A_{\sigma_k} \cong$ vector space of basic eigenfunctions of $\Delta^M$ with eigenvalue $\lambda_k$. Note also that if $\mathscr{F}$ is given by the fibres of a harmonic Riemannian submersion $f\colon M \to N$ then $\Omega_B^0 = \Omega^0(N)$ and $\mathrm{Spec}_B = \mathrm{Spec}\,(N)\setminus\{0\}$, $K = \infty$ and $\dim A_{\sigma_k} =$ multiplicity of $\lambda_k$ as an eigenvalue for $\Delta^N$.

3.4. EXAMPLE. If $\mathscr{F}$ is defined by the fibres of the Hopf map $f\colon S^{2n+1} \to \mathbb{C}P^n$ then $q = 2n$, $\rho_\nabla = \rho^{\mathbb{C}P^n} = 2(n+1)\mathrm{id}$; that is,

$$\lambda = 4(n+1),\ \mathrm{Spec}_B = \{\lambda_k = 4k(n+k)\}_{k=1}^\infty$$

and

$$\sum = \left\{\sigma_k = \frac{n+1}{k(k+n)} - 1\right\}_{k=1}^\infty.$$

As $\Sigma \leqslant 0$, for $n \geqslant 2$, every transversal conformal automorphism is Killing. On the other hand, for $n = 1$, i.e. $q = 2$, we have $1 - (2/q) = \sigma_1 = 0$ and $\dim A_0 = 3$.

3.5. EXAMPLE. Let $\mathscr{F}$ be defined by the fibres of a harmonic Riemannian submersion $f\colon M \to S^n$, $n \geqslant 2$. (A large class of such maps are constructed in [1].) Then $q = n$, $\rho_V = \rho S^n = (n-1)\,\mathrm{id}$; that is,

$$\lambda = 2(n-1),\ \mathrm{Spec}_B = \{\lambda_k = k(k+n-1)\}_{k=1}^{\infty}$$

and

$$\sum = \left\{\sigma_k = \frac{2(n-1)}{k(k+n-1)} - 1\right\}_{k=1}^{\infty}.$$

(As $\sigma_1 = 1 - (2/q)$ the bounds for $\Sigma$ in Theorem 3.1 are the best possible.) For $n \neq 2, 0 \notin \Sigma$ and so every transversal Jacobi automorphism is Killing. For $n = 2$, $\sigma_1 = 0$ and again $\dim A_0 = 3$.

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