REMARKS ON WENTE'S EXAMPLE
OF AN IMMERSED TORUS IN $E^3$

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ABSTRACT. In 1984, H. Wente gave an example of an immersed torus in $E^3$ with constant mean curvature thereby resolving the so-called Hopf conjecture. In this paper, the local behaviour of the Gaussian curvature $K$ near its zero set is studied. This gives rise to a solution of the Hopf problem provided an assumption on the asymptotic behaviour of $K$ near its zero set is made.

1. Hopf's problem. In 1950, Heinz Hopf proved that a closed orientable surface of genus zero immersed in Euclidean 3-space $E^3$ with constant mean curvature is a round sphere $S^2$ [7,8], i.e., an isometrically embedded sphere. He asked if the condition on the genus could be removed. In 1955, A. D. Aleksandrov showed that for embedded surfaces it could [2,7], and he conjectured that this was valid for immersions as well. It was not until 1984 when H. Wente [11] gave a striking example of an immersed torus in $E^3$ with constant mean curvature that this problem was resolved. The construction required a detailed analysis of the sinh-Gordon equation, i.e., the Gauss equation (cf. also [1]). Note that the Gauss map of such an immersion is harmonic [10]. However, it is not holomorphic, and by a result of J. Eells and J. C. Wood [6], it has degree zero.

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In this paper, the local behaviour of the Gaussian curvature $K$ near its zero set $Z$ is analyzed. This leads to a solution of the Hopf problem provided an assumption on the asymptotic behaviour of $K$ near $Z$ is made.

**Theorem 1.** Let $M$ be a closed surface immersed in $E^3$ with constant mean curvature, and let $\gamma$ be the function defined by

$$\gamma(s) = s \cdot \int_{\{K^2 \leq 1/s\}} AK^2, \quad s > 0,$$

where $\{K^2 \leq 1/s\}$ is the set of all $x \in M$ such that $K^2(x) \leq 1/s$. Then, either $M$ is a round sphere (and therefore $\gamma = 0$) or

$$\lim_{s \to \infty} \gamma(s) = \infty,$$

and the derivative of $\gamma$ is eventually strictly positive.

**Corollary.** Let $M$ be a closed surface immersed in $E^3$ with constant mean curvature. Then, if

$$\int_{\{K^2 \leq 1/s\}} AK^2 = O(\frac{1}{s}) \quad \text{for} \quad s \to \infty,$$

$M$ is a round sphere.

Theorem 1 shows that Wente's example is indeed very special. It's proof is a consequence of a method developed by S.-S. Chern and S. I. Goldberg in
an analysis of the sinh-Gordon equation, and a classical method of
Nevanlinna theory which uses an appropriate exhaustion function on \( \Omega \).
Details will appear elsewhere.

2. The superharmonicity of \( \log K^2 \), the critical points of \( K \),
and Wente's example. Let \( M \) be a closed surface immersed in \( E^3 \) with
constant mean curvature which is normalized to be \( 1/2 \). Since this latter
condition is expressed by an absolutely elliptic equation \([7]\), it follows from
Bernstein's theorem that \( M \) is analytic in \( E^3 \). Hence, all data derived from
the metric, such as the Gaussian curvature \( K \), are (real) analytic. In
particular, \( L = \text{Zero}(K) \) is an analytic set in \( M \), and so applying the
Weierstrass Preparation Theorem \([9]\), it consists of finitely many analytic
curves.

The behaviour of the function \( \gamma \) in (1) is determined by a certain
nonnegative (analytic) scalar invariant \( C \) of the Gauss map \( \varphi : M \to S^2 \)
of the immersion of \( M \) into \( E^3 \) given in \([4]\). In fact, it turns out that
\( C = |\psi \beta|^2/4 \), where \( \beta \) is the second fundamental form of the immersion, and
\( \psi \beta \) is its covariant derivative. Note that \( C = 0 \) if and only if \( M \) is a
round sphere.

In the sequel, it is assumed that \( M \neq S^2 \). The following global formulas
will be required:

\[
(3) \quad \frac{1}{2} \Delta K^2 = (1 - 4K)(2C - K^2) - C
\]

and on \( \Omega \): \( \Omega \):

\[
(4) \quad \frac{1}{2} \Delta \log K^2 = -(1 - 4K) - \frac{C}{K^2}.
\]
The latter says that \( \log k^2 \) is superharmonic away from \( Z \) (see [4], p. 143) since \( 1 - 4K \geq 0 \) (with equality at the umbilics), which is an important fact in the proof of Theorem 1.

Away from umbilics, \( M \) is locally given by a conformal representation \( F \) which is determined by a solution of the sinh-Gordon equation

\[
\Delta \omega + \frac{1}{2} \sinh 2\omega = 0,
\]

where \( e^{-2\omega} \) corresponds, via \( F \), to the (positive) difference \( \lambda_2 - \lambda_1 \) of the principal curvatures [1]. Since \( \lambda_1 + \lambda_2 = 1 \), the Gaussian curvature \( K \) corresponds to \( (1 - e^{-4\omega})/4 \).

A point \( z_0 \in Z \) is said to be smooth if there is a neighborhood \( U \) of \( z_0 \) such that \( U \cap Z \) is a single analytic arc. A nonsmooth point \( z_0 \in Z \) is said to be a \( C^1 \)-meeting point of \( q \) general folds if a neighborhood \( U \) of \( z_0 \) is \( C^1 \) diffeomorphic to a neighborhood of the origin in \( E^2 \) with \( U \cap Z \) corresponding to \( q(>1) \) line segments meeting at the origin (see [12] and [5, p. 53]).

**THEOREM 2.** If \( M \neq S^2 \), then \( C \) and \( dK \) do not vanish at the smooth points of \( Z \).

The proof follows from the sinh-Gordon equation, the maximum principle for subharmonic functions, and the superharmonicity of \( K \) on the set \( \{K > 0\} \).

**Remarks.** (a) The converse of Theorem 2 is also true and boils down to the implicit function theorem, namely, if \( z_0 \in Z \) with \( C(z_0) \neq 0 \), then \( z_0 \) is a smooth point of \( Z \). For, \( |dK|^2 = C \) on \( Z \).
(b) Theorem 2 implies that the critical points of $K$ on $Z$ are isolated on $Z$. In fact, they are also isolated on $M$.

**THEOREM 3.** If $M \neq S^2$, any nonsmooth point $z_0 \in Z$ is a $C^1$-meeting point of an even number of general folds.

**Proof.** $K$ is the ratio of surface elements with respect to the Gauss map $\varphi : M \to S^2$. Thus, $Z$ is the set of singular points of $\varphi$. We can then use the classification of such points given by J. C. Wood [12]. This, together with the nature of the sinh-Gordon equation yields the result.

**Remark.** In Wente's example, $Z$ is a union of figure eights, and the nonsmooth points are meeting points of two general folds. Moreover, $C$ vanishes only at the nonsmooth points, and the condition $K^2 < 2C$ in [4. Proposition 3.4] is not satisfied on $Z$.

3. **Completion of the proof of Theorem 1.** If $M \neq S^2$, then by Theorem 2, $C$ does not vanish identically on $Z$. Formula (2) then follows from

**PROPOSITION 1.** If $C$ is not identically zero on $Z$, then

$$\lim_{s \to \infty} \gamma(s) = \infty.$$
Proof. By (3).

\[ \gamma(s) = 2s \cdot \int_{\{K^2 \leq 1/s\}} (1 - 8K)C - 2s \cdot \int_{\{K^2 \leq 1/s\}} (1 - 4K)K^2 \]

\[ \geq s \cdot \int_{\{K^2 \leq 1/s\}} C \text{ for large } s. \]

To show that the r.h.s. diverges, we first note that, by hypothesis, \( C \) does not vanish on an arc \( \Gamma \subset \mathbb{C} \). The proposition is then a consequence of the following elementary lemma applied to a tubular neighborhood of \( \Gamma \) in \( M \).

**Lemma 2.** Let \( u \) be a \( C^1 \) function on \([0,1] \times [-1,1]\) with \( \text{Zero}(u) = [0,1] \times \{0\} \). Then, for some \( a > 0 \),

\[ \text{Area}(\{|u| \leq \varepsilon\}) \geq a \varepsilon \]

uniformly for \( \varepsilon \to 0 \).

The last statement in Theorem 1 is the content of

**Proposition 2.** Given \( M \) as in Theorem 1, the function \( \gamma \) is eventually nondecreasing. If the derivative of \( \gamma \) vanishes on a divergent sequence \( s_n \to \infty \), then \( M \) is a round sphere.

For the proof we shall require the following two lemmas, where \( M \) is considered as a compact Riemann surface.

**Lemma 3.** Let \( \tau \) be an exhaustion function on \( M \mathbb{C} \). Then, for \( r \)

large.
\[
\int_{\{r \leq r_0\}} \Delta \log k^2 = \frac{d}{dr} \int_{\{r = r\}} \log k^2 \cdot d^c r.
\]

where \( d^c = 1(\mathbf{1} - \theta) \) (see [3, p. 18] for notation).

**Lemma 4.** The function \( k^2 \) has no critical points near \( Z \).

**Proof of Proposition 2.** If \( N \neq S^2 \), then by Lemma 4, we can choose \( \tau = \log \log (1/k^2) \) near \( Z \). We may then extend it to an exhaustion function \( \tau \) on the whole of \( M \setminus Z \). Along \( \{\tau = r\} \), for \( r \) large,

\[
d^c \tau = d^c (\log \log \frac{1}{k^2}) = \frac{1}{\log k^2} \frac{d^c k^2}{k^2}.
\]

and so (5) becomes

\[
\int_{\{r \leq r_0\}} \Delta \log k^2 = \frac{d}{dr} \left( e^{r} \int_{\{r = r\}} d^c k^2 \right).
\]

By a change of variable, \( s = e^r \), we obtain, for \( s \) large,

\[
\frac{1}{s \log s} \int_{\{k^2 \leq 1/s\}} \Delta \log k^2 = \frac{d}{ds} \left( s \int_{\{k^2 \leq 1/s\}} d^c k^2 \right)
= -\frac{d}{ds} \left( s \int_{\{k^2 \leq 1/s\}} \Delta k^2 \right) = -\gamma'(s),
\]

where in the second equality Stokes' theorem is used. Now, by the

superharmonicity of \( \log k^2 \), \( \gamma \) is eventually nondecreasing. Finally, if
\( \gamma'(a_n) = 0 \) for some sequence \( a_n \to \infty \) then, again by the superharmonicity of \( \log k^2 \) and (6), it follows that \( A \log k^2 = 0 \) on \( MN \). By (4) this means that \( C = 0 \) everywhere, and so \( N \) is a round sphere which is a contradiction.

REFERENCES


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