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REMARKS ON WENTE'S EXAMPLE OF AN IMMERSED TORUS IN E^3

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ABSTRACT. In 1984, H. Wente gave an example of an immersed torus in E^3 with constant mean curvature thereby resolving the so-called Hopf conjecture. In this paper, the local behaviour of the Gaussian curvature K near its zero set is studied. This gives rise to a solution of the Hopf problem provided an assumption on the asymptotic behaviour of K near its zero set is made.

1. Hopf's problem. In 1950, Heinz Hopf proved that a closed orientable surface of genus zero immersed in Euclidean 3-space E^3 with constant mean curvature is a round sphere S^2 [7,8], i.e., an isometrically embedded sphere. He asked if the condition on the genus could be removed. In 1955, A. D. Aleksandrov showed that for embedded surfaces it could [2,7], and he conjectured that this was valid for immersions as well. It was not until 1984 when H. Wente [11] gave a striking example of an immersed torus in E^3 with constant mean curvature that this problem was resolved. The construction required a detailed analysis of the sinh-Gordon equation, i.e., the Gauss equation (cf. also [1]). Note that the Gauss map of such an immersion is harmonic [10]. However, it is not holomorphic, and by a result of J. Eells and J. C. Wood [6], it has degree zero.

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In this paper, the local behaviour of the Gaussian curvature K near its zero set Z is analyzed. This leads to a solution of the Hopf problem provided an assumption on the asymptotic behaviour of K near Z is made.

THEOREM 1. *Let* M *be a closed surface immersed in* E^3 *with constant mean curvature, and let* γ *be the function defined by*

$$\gamma(s) = s \cdot \int_{\{K^2 \leqq 1/s\}} \Delta K^2, \quad s > 0. \tag{1}$$

where $\{K^2 \leqq 1/s\}$ *is the set of all* $x \in M$ *such that* $K^2(x) \leqq 1/s$. *Then, either* M *is a round sphere (and therefore* $\gamma = 0$*) or*

$$\lim_{s \to \infty} \gamma(s) = \infty, \tag{2}$$

and the derivative of γ *is eventually strictly positive*.

COROLLARY. *Let* M *be a closed surface immersed in* E^3 *with constant mean curvature*. *Then, if*

$$\int_{\{K^2 \leqq 1/s\}} \Delta K^2 = O(\tfrac{1}{s}) \quad \text{for} \quad s \to \infty,$$

M *is a round sphere*.

Theorem 1 shows that Wente's example is indeed very special. It's proof is a consequence of a method developed by S.-S. Chern and S. I. Goldberg in

[4], an analysis of the sinh-Gordon equation, and a classical method of Nevanlinna theory which uses an appropriate exhaustion function on $M \setminus Z$. Details will appear elsewhere.

2. The superharmonicity of $\log K^2$, the critical points of K, and Wente's example. Let M be a closed surface immersed in E^3 with constant mean curvature which is normalized to be $1/2$. Since this latter condition is expressed by an absolutely elliptic equation [7], it follows from Bernstein's theorem that M is analytic in E^3. Hence, all data derived from the metric, such as the Gaussian curvature K, are (real) analytic. In particular, $Z = \mathrm{Zero}(K)$ is an analytic set in M, and so applying the Weierstrass Preparation Theorem [9], it consists of finitely many analytic curves.

The behaviour of the function γ in (1) is determined by a certain nonnegative (analytic) scalar invariant C of the Gauss map $\varphi : M \to S^2$ of the immersion of M into E^3 given in [4]. In fact, it turns out that $C = |\nabla\beta|^2/4$, where β is the second fundamental form of the immersion, and $\nabla\beta$ is its covariant derivative. Note that $C = 0$ if and only if M is a round sphere.

In the sequel, it is assumed that $M \neq S^2$. The following global formulas will be required:

$$\frac{1}{2}\Delta K^2 = (1 - 4K)(2C - K^2) - C \tag{3}$$

and on $M \setminus Z$,

$$\frac{1}{2}\Delta \log K^2 = -(1 - 4K) - \frac{C}{K^2}\,. \tag{4}$$

The latter says that $\log K^2$ *is superharmonic away from* Z (see [4], p. 143) since $1 - 4K \geqq 0$ (with equality at the umbilics), which is an important fact in the proof of Theorem 1.

Away from umbilics, M is locally given by a conformal representation F which is determined by a solution of the sinh-Gordon equation

$$\Delta\omega + \frac{1}{2}\sinh 2\omega = 0,$$

where $e^{-2\omega}$ corresponds, via F, to the (positive) difference $\lambda_2 - \lambda_1$ of the principal curvatures [1]. Since $\lambda_1 + \lambda_2 = 1$, *the Gaussian curvature* K *corresponds to* $(1 - e^{-4\omega})/4$.

A point $z_0 \in Z$ is said to be *smooth* if there is a neighborhood U of z_0 such that $U \cap Z$ is a single analytic arc. A nonsmooth point $z_0 \in Z$ is said to be a C^1-*meeting point of* q *general folds* if a neighborhood U of z_0 is C^1 diffeomorphic to a neighborhood of the origin in E^2 with $U \cap Z$ corresponding to $q(>1)$ line segments meeting at the origin (see [12] and [5, p. 53]).

THEOREM 2. *If* $M \neq S^2$, *then* C *and* dK *do not vanish at the smooth points of* Z.

The proof follows from the sinh-Gordon equation, the maximum principle for subharmonic functions, and the superharmonicity of K on the set $\{K > 0\}$.

Remarks. (a) The converse of Theorem 2 is also true and boils down to the implicit function theorem, namely, *if* $z_0 \in Z$ *with* $C(z_0) \neq 0$, *then* z_0 *is a smooth point of* Z. For, $|dK|^2 = C$ on Z.

(b) Theorem 2 implies that the critical points of K on Z are isolated on Z. In fact, they are also isolated on M.

THEOREM 3. If $M \neq S^2$, any nonsmooth point $z_0 \in Z$ is a C^1-meeting point of an even number of general folds.

Proof. K is the ratio of surface elements with respect to the Gauss map $\varphi : M \to S^2$. Thus, Z is the set of singular points of φ. We can then use the classification of such points given by J. C. Wood [12]. This, together with the nature of the sinh-Gordon equation yields the result.

Remark. In Wente's example, Z is a union of figure eights, and the nonsmooth points are meeting points of two general folds. Moreover, C vanishes only at the nonsmooth points, and the condition $K^2 \geq 2C$ in [4, Proposition 3.4] is not satisfied on Z.

3. Completion of the proof of Theorem 1. If $M \neq S^2$, then by Theorem 2, C does not vanish identically on Z. Formula (2) then follows from

PROPOSITION 1. If C is not identically zero on Z, then

$$\lim_{s \to \infty} \gamma(s) = \infty.$$

Proof. By (3),

$$\gamma(s) = 2s \cdot \int_{\{K^2 \leqq 1/s\}} (1 - 8K)C - 2s \cdot \int_{\{K^2 \leqq 1/s\}} (1 - 4K)K^2$$

$$\geqq s \cdot \int_{\{K^2 \leqq 1/s\}} C \quad \text{for large } s.$$

To show that the r.h.s. diverges, we first note that, by hypothesis, C does not vanish on an arc $\Gamma \subset Z$. The proposition is then a consequence of the following elementary lemma applied to a tubular neighborhood of Γ in M.

LEMMA 2. *Let* u *be a* C^1 *function on* $[0,1] \times [-1,1]$ *with* $\text{Zero}(u) = [0,1] \times \{0\}$. *Then, for some* $a > 0$,

$$\text{Area}\{|u| \leqq \varepsilon\} \geqq a \cdot \varepsilon$$

uniformly for $\varepsilon \to 0$.

The last statement in Theorem 1 is the content of

PROPOSITION 2. *Given* M *as in Theorem 1, the function* γ *is eventually nondecreasing. If the derivative of* γ *vanishes on a divergent sequence* $s_n \to \infty$, *then* M *is a round sphere*.

For the proof we shall require the following two lemmas, where M is considered as a compact Riemann surface.

LEMMA 3. *Let* τ *be an exhaustion function on* $M \setminus Z$. *Then, for* r *large*,

(5) $$\int_{\{\tau \leqq r\}} \Delta \log K^2 = \frac{d}{dr} \int_{\{\tau = r\}} \log K^2 \cdot d^c\tau,$$

where $d^c = i(\bar{\partial} - \partial)$ (*see* [3, p. 18] *for notation*).

LEMMA 4. *The function* K^2 *has no critical points near* Z.

Proof of Proposition 2. If $M \neq S^2$, then by Lemma 4, we can choose $\tau = \log \log(1/K^2)$ near Z. We may then extend it to an exhaustion function τ on the whole of $M \setminus Z$. Along $\{\tau = r\}$, for r large,

$$d^c\tau = d^c(\log \log \frac{1}{K^2}) = \frac{1}{\log K^2} \frac{d^cK^2}{K^2},$$

and so (5) becomes

$$\int_{\{\tau \leqq r\}} \Delta \log K^2 = \frac{d}{dr} \{e^{e^r} \int_{\{\tau = r\}} d^c K^2\}.$$

By a change of variable, $s = e^{e^r}$, we obtain, for s large,

(6) $$\frac{1}{s \log s} \int_{\{K^2 \geqq 1/s\}} \Delta \log K^2 = \frac{d}{ds} \{s \cdot \int_{\{K^2 = 1/s\}} d^c K^2\}$$

$$= -\frac{d}{ds} \{s \cdot \int_{\{K^2 \leqq 1/s\}} \Delta K^2\} = -\gamma'(s),$$

where in the second equality Stokes' theorem is used. Now, by the superharmonicity of $\log K^2$, γ is eventually nondecreasing. Finally, if

$\gamma'(s_n) = 0$ for some sequence $s_n \to \infty$ then, again by the superharmonicity of $\log K^2$ and (6), it follows that $\Delta \log K^2 = 0$ on $M\setminus Z$. By (4) this means that $C = 0$ everywhere, and so M is a round sphere which is a contradiction.

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