

ON NONRIGIDITY OF HARMONIC MAPS INTO SPHERES

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ABSTRACT. This note studies nonrigidity of equivariant harmonic maps $f: M \rightarrow S^n$ of a Riemannian homogeneous space M into the Euclidean n -sphere S^n via representation theory applied to the induced module structure on \mathbf{R}^n and, for specific M , produces (divergence-free) Jacobi fields along f which do not come from isometric deformations of f on the range.

1. Introduction. This note, originating from the works of R. T. Smith [5] and Do Carmo and Wallach [3], continues the studies [7–9] on rigidity of a harmonic map $f: M \rightarrow S^n$, $n \geq 2$, with constant energy density $e(f) = \lambda/2$, $\lambda \in \text{Spec}(M)$, of a compact Riemannian manifold M into the Euclidean n -sphere S^n via the (finite-dimensional) vector space $K(f)$ of all divergence-free Jacobi fields along f . Recall [4] that f is harmonic iff $\Delta^M f = \lambda \cdot f$. Furthermore, by translating tangent vectors of $S^n \subset \mathbf{R}^{n+1}$ to the origin of \mathbf{R}^{n+1} , a vector field v along f gives rise to a vector function $\hat{v}: M \rightarrow \mathbf{R}^{n+1}$, $\langle \hat{v}, f \rangle = 0$. Then [9], $v \in K(f)$ iff $\Delta^M \hat{v} = \lambda \cdot \hat{v}$.

The vector space $\text{so}(n+1) \circ f$ of infinitesimal isometric deformations of f is a linear subspace of $K(f)$ and f is said to be *rigid* if $\text{so}(n+1) \circ f = K(f)$. The importance of $K(f)$ is shown by the generalized Do Carmo-Wallach classification theorem [9] which states that, for oriented (isotropy) irreducible Riemannian homogeneous M , the equivalence classes of full harmonic maps $f: M \rightarrow S^n$ with $e(f) = \lambda/2$ can be parametrized by a compact convex body lying in $K(f_\lambda)/\text{so}(V_\lambda) \circ f_\lambda$, where $f_\lambda: M \rightarrow S^{n(\lambda)}$ ($=$ unit sphere in V_λ) is a standard minimal immersion given by an orthonormal base in the eigenspace V_λ corresponding to λ . Though there is no immediate generalization of this classification to arbitrary Riemannian homogeneous M the object of this note is to show that the occurrence of rigidity is rare. In §2, we reformulate the condition of rigidity in terms of representation theory. This is then applied in §3 to prove nonrigidity of harmonic maps in various instances.

2. Properties of rigid harmonic maps.

THEOREM 1. *Let $M = G/K$ be a compact naturally reductive Riemannian homogeneous space (with base point $o = \{K\}$) and $f: M \rightarrow S^n$, $n \geq 2$, a full rigid harmonic map with $e(f) = \lambda/2$, $\lambda \in \text{Spec}(M)$. Then (a) f is equivariant with respect to an orthogonal G -module structure on \mathbf{R}^{n+1} ; (b) \mathbf{R}^{n+1} is an irreducible G -submodule of V_λ ;*

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(c) $\dim \text{Fix}(K, \mathbf{R}^{n+1}) = 1, 2$ or 4 and \mathbf{R}^{n+1} is a real, complex or quaternionic G -module, accordingly; (d) $S^2(\mathbf{R}^{n+1})$ (= symmetric square of \mathbf{R}^{n+1}) is the sum of irreducible G -submodules which are class 1 for (G, K) .

PROOF. (a) See [9].

(b) Let $\mathbf{R}^{n+1} = V \oplus W$ be a nontrivial orthogonal G -invariant decomposition. Then $f = (g, h)$, where $g: M \rightarrow V$ and $h: M \rightarrow W$ are equivariant, in particular, $\|g\|^2$ and $\|h\|^2$ ($= 1 - \|g\|^2$) are constants. As f is full we have $\|g\| = \cos t$ and $\|h\| = \sin t$ for some $0 < t < \pi/2$. The vector function $\hat{v} = (-\tan t \cdot g, \cot t \cdot h): M \rightarrow \mathbf{R}^{n+1}$ gives rise to an element $v \in K(f)$ which, by fullness of f , does not belong to $\text{so}(n+1) \circ f$. Thus, \mathbf{R}^{n+1} is irreducible and mapping the i th base vector to the i th component of f induces a G -module monomorphism $\mathbf{R}^{n+1} \rightarrow V_\lambda$.

(c) Setting $F = \text{Fix}(K, \mathbf{R}^{n+1})$, we first note that the orthogonal complement $f(o)^\perp$ of $f(o)$ in F corresponds bijectively to the set of G -invariant vector fields along f . Assuming that $\dim F \geq 2$, let $\hat{v}_o \in f(o)^\perp$, $\|\hat{v}_o\| = 1$, and denote by v the corresponding G -invariant vector field along f . Then [8] $v \in K(f)$ and so $\hat{v} = J \cdot f$, where $J \in \text{so}(n+1)$. We claim that J is a complex structure on \mathbf{R}^{n+1} , i.e. $J^2 = -I_{n+1}$ (I_{n+1} = identity of \mathbf{R}^{n+1}). Indeed, setting $\hat{u} = (J^2 + I_{n+1}) \cdot f$, we have $\langle \hat{u}, f \rangle = 0$ and so \hat{u} induces a vector field u along f which automatically belongs to $K(f) = \text{so}(n+1) \circ f$. Thus, by fullness of f , $J^2 + I_{n+1}$ is skew and hence zero. By G -invariance of v , J commutes with the action of G on \mathbf{R}^{n+1} . For $\dim F \geq 3$ we proceed analogously by selecting an orthonormal base $\{\hat{v}_o^i\}_{i=1}^p \subset f(o)^\perp$, $\dim F = p + 1$, and considering the corresponding anticommuting family $\{J^i\}_{i=1}^p$ of linearly independent skew-symmetric complex structures on \mathbf{R}^{n+1} . By simple argument (involving $J^1 \cdot J^2$), it follows that $p \geq 3$. On the other hand, endowing \mathbf{R}^{n+1} with the complex structure J^1 and applying Schur's lemma to $J^2 J^3$, we have $J^2 J^3 = \alpha I_{n+1} + \beta J^1$, $\alpha, \beta \in \mathbf{R}$. By simple algebra, $\alpha = 0$ and $\beta = \pm 1$. The rest is clear.

(d) The G -submodule $W_f = \text{span}\{f(x)^2 | x \in M\} \subset S^2(\mathbf{R}^{n+1})$ splits into the sum of class 1 subrepresentations for (G, K) . We claim that $W_f = S^2(\mathbf{R}^{n+1})$. Indeed, setting $B \in W_f^\perp \subset S^2(\mathbf{R}^{n+1})$ the vector function $\hat{w} = B \cdot f$ satisfies $\langle \hat{w}, f \rangle = 0$ and hence gives rise to an element w of $K(f)$. By rigidity, B is skew and hence zero.

REMARKS. (1) If K acts on $T_o(M)$ (via the isotropy representation) without nonzero fixed vectors (e.g. if M is irreducible) then $\dim F = \dim \text{Fix}(K, \mathbf{R}^{n+1}) = 1$. Indeed, assume that $\hat{v}_o \in f(o)^\perp$, $\|\hat{v}_o\| = 1$, exists and consider the corresponding G -invariant vector field v and complex structure J on \mathbf{R}^{n+1} . Choose a Killing vector field X on M such that $f_*(X) = Y \circ f$ for a nonzero $Y \in \text{so}(n+1)$. Setting $\hat{u} = J \cdot Y \cdot f$, as v is orthogonal to $\text{im}(f) \subset S^n$, we have $\langle \hat{u}, f \rangle = -\langle Y \cdot f, J \cdot f \rangle = -\langle f_*(X), \hat{v} \rangle = 0$. Hence \hat{u} induces a vector field $u \in K(f)$ along f and, by rigidity, $J \cdot Y = Z \in \text{so}(n+1)$. Taking transposes, it follows that J and Y anticommute. On the other hand, J , being a G -module automorphism, commutes with Y , i.e. $J \cdot Y = 0$. This implies $Y = 0$, a contradiction.

(2) As Professor Joseph A. Wolf has informed the author, for $M = G/K$ compact irreducible symmetric with $\text{rank } M = \text{rank } G$, the symmetric square of every irreducible component of V_λ , $\lambda \in \text{Spec}(M)$, is the sum of class 1 subrepresentations for (G, K) .

A partial converse of Theorem 1 is given as follows:

THEOREM 2. *Let $M = G/K$ be a compact irreducible Riemannian homogeneous space and $\lambda \in \text{Spec}(M)$ such that V_λ is irreducible, the K -module $V_\lambda|K$ has multiplicity 1 decomposition into irreducible components and $S^2(V_\lambda)$ is the sum of class 1 subrepresentations for (G, K) . Then the standard minimal immersion $f_\lambda: M \rightarrow S^{n(\lambda)}$ is rigid.*

PROOF. Given $v \in K(f_\lambda)$ we have $\hat{v} = B \cdot f_\lambda$ for some matrix B since the components of f_λ form a basis in V_λ . We may assume that B is symmetric (by splitting B into symmetric and skew-symmetric parts if necessary). As $0 = \langle \hat{v}, f_\lambda \rangle = \langle B \cdot f_\lambda, f_\lambda \rangle$, we have $B \in W_{f_\lambda}^\perp$, where $W_{f_\lambda} = \text{span}\{f_\lambda(x)^2 | x \in M\} \subset S^2(V_\lambda)$. On the other hand, as $V_\lambda|K$ has multiplicity 1 decomposition, we can apply the argument of Do Carmo and Wallach in the proof of 4.2. Lemma in [3, pp. 50–51], and it follows that W_{f_λ} is the sum of all class 1 subrepresentations of (G, K) in $S^2(V_\lambda)$. Then, by hypothesis, $W_{f_\lambda} = S^2(V_\lambda)$ which implies that $B = 0$.

REMARK. For M rank 1 symmetric and $\lambda \in \text{Spec}(M)$ the first two assumptions on V_λ are automatically satisfied (cf. [1, 6]) and so f_λ is rigid iff $S^2(V_\lambda)$ is the sum of class 1 subrepresentations for (G, K) . In particular, for

$$(G, K) = (\text{SO}(m+1), \text{SO}(m)),$$

f_λ is rigid iff $m = 2$ (Calabi's rigidity [2]) or f_λ is of degree 1 (isometry [9]). For $(G, K) = (SU(m+1), S(U(m) \times U(1)))$, $m \geq 2$, and $(G, K) = (\text{Sp}(3), \text{Sp}(2) \times \text{Sp}(1))$, any standard minimal immersion f_λ of degree ≥ 4 is nonrigid by the results of Urakawa in [10].

3. Applications.

THEOREM 3. *Let $f: M_1 \times M_2 \rightarrow S^n$ be a full harmonic map with constant energy density of the Riemannian product of compact naturally reductive Riemannian homogeneous spaces M_1 and M_2 . If f does not factor through the canonical projections $\pi_i: M_1 \times M_2 \rightarrow M_i$, $i = 1, 2$, then f is nonrigid.*

PROOF. Assuming that f is rigid, by Theorem 1, f is equivariant with respect to an irreducible $G_1 \times G_2$ -module structure on \mathbf{R}^{n+1} , where the compact Lie group G_i acts transitively on M_i by isometries. Then $\mathbf{R}^{n+1} = V_1 \otimes V_2$, where V_i is an irreducible G_i -module over \mathbf{R}, \mathbf{C} , or \mathbf{H} according as $\dim \text{Fix}(K, \mathbf{R}^{n+1}) = 1, 2$ or 4 . In particular, $f(o) = v_1 \otimes v_2$, $v_i \in V_i$, and, by equivariance, $f(x_1, x_2) = f_1(x_1) \otimes f_2(x_2)$, $x_i \in M_i$, where $f_i: M_i \rightarrow V_i$. As f does not factor through π_i , we have $\dim V_i \geq 2$ so that we can select nonzero matrices $A_i \in \text{so}(V_i)$. Define the vector function $\hat{u}: M_1 \times M_2 \rightarrow \mathbf{R}^{n+1}$ by $\hat{u}(x_1, x_2) = (A_1 \otimes A_2, (f_1(x_1) \otimes f_2(x_2)))$, $x_i \in M_i$. Then $\langle \hat{u}, f \rangle = 0$ and \hat{u} induces a vector field $u \in K(f)$ along f . By rigidity and fullness of f , $A_1 \otimes A_2$ is skew, a contradiction.

EXAMPLE. As an easy computation (involving the spectrum of $S^2 \times S^3$) shows, the canonical projection $\pi_1: S^2 \times S^3 \rightarrow S^2$ followed by the Veronese map $f_{\lambda_2}: S^2 \rightarrow S^4$ is rigid.

THEOREM 4. *Given a compact naturally reductive Riemannian homogeneous space M and $\lambda \in \text{Spec}(M)$, the number of equivalence classes of full rigid harmonic maps $f: M \rightarrow S^n$ with $e(f) = \lambda/2$ does not exceed the number of inequivalent irreducible components of V_λ .*

PROOF. Given an irreducible component \mathbf{R}^{n+1} in V_λ we have to show that any two full rigid equivariant harmonic maps $f, f': M \rightarrow S^n$ with $e(f) = e(f') = \lambda/2$ are equivalent. This is done by using the uniqueness of K -fixed vectors over \mathbf{R}, \mathbf{C} or \mathbf{H} .

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