ON NATURALLY REDUCTIVE HOMOGENEOUS SPACES HARMONICALLY EMBEDDED INTO SPHERES

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1. *Introduction and preliminaries*

This note continues earlier studies [**9**, **10**] concerning rigidity properties of harmonic maps into spheres. Given a harmonic map $f : M \to S^n$, $n \geqslant 2$ [**5**] of a compact Riemannian manifold M into the Euclidean n-sphere S^n the (finite dimensional) vector space $K(f)$ of all divergence free Jacobi fields along f [**7**] contains the vector space of infinitesimal isometric deformations $so(n+1) \circ f$, where $so(n+1)$ is identified with the Lie algebra of Killing vector fields on S^n [**8**, **9**]. Recall that the harmonic map f is said to be *infinitesimally rigid* if $so(n+1) \circ f = PK(f)$, where $PK(f) \subset K(f)$ is the projectable part, that is

$$PK(f) = \{v \in K(f) \mid v_x = v_{x'} \text{ whenever } f(x) = f(x'),\ x, x' \in M\}.$$

Any map $f : M \to S^n$ can be considered, via the inclusion $S^n \subset \mathbb{R}^{n+1}$, as a vector function $f : M \to \mathbb{R}^{n+1}$ with components $(f^1, \ldots, f^{n+1})$ such that

$$\langle f, f \rangle = \sum_{i=1}^{n+1} (f^i)^2 = 1.$$

The map f is harmonic if and only if the corresponding vector function satisfies the equation

$$\Delta^M f = 2e(f) \cdot f, \tag{1}$$

where Δ^M is the Laplacian on M and $e(f)$ denotes the energy density of f. (Here and in what follows we use the notation (for example the sign conventions) of [**4**] and this serves as a general reference for harmonic maps as well.) Further, by translating tangent vectors of $S^n \subset \mathbb{R}^{n+1}$ to the origin, a vector field v along f (that is, a section of the pull-back bundle $f^*(T(S^n))$) gives rise to a vector function $\hat{v} : M \to \mathbb{R}^{n+1}$ with the obvious property that $\langle f, \hat{v} \rangle = 0$. Then [**6**] $v \in K(f)$ if and only if

$$\Delta^M \hat{v} = 2e(f) \cdot \hat{v}; \tag{2}$$

that is, $K(f)$ can be identified with the vector space of all solutions $\hat{v} : M \to \mathbb{R}^{n+1}$ of (2) which satisfy the linear constraint $\langle f, \hat{v} \rangle = 0$.

The purpose of this paper is to study infinitesimal rigidity of harmonic maps $f : M \to S^n$ when $M = G/H$ is a naturally reductive Riemannian homogeneous space. By a result of [**10**], any full infinitesimally rigid harmonic embedding

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$f : G/H \to S^n$ with constant energy density is equivariant, that is there exists a (necessarily unique) monomorphism $\rho : G \to SO(n+1)$ such that f is equivariant with respect to ρ. To obtain restrictive conclusions on the behaviour of infinitesimally rigid equivariant harmonic maps we consider equivariant vector fields along f which, as is proved in Section 2, belong to $K(f)$. In this case, under fairly general assumptions, we show that the product of $\rho(G)$ with its centralizer in $SO(n+1)$ acts transitively on S^n and then a description of such groups given by A. Borel [2, 3] is exploited in Section 3; this yields the result that infinitesimal rigidity is not generally present.

Throughout this note all objects are smooth, that is, of class C^∞.

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2. *Equivariant vector fields along harmonic maps*

In what follows we consider a compact Lie group G and a closed subgroup $H \subset G$ such that the canonical action τ of G on G/H (induced by left translations) is effective, that is H does not contain proper closed subgroups which are normal in G. We endow G with a fixed biinvariant metric. Denoting by $\mathfrak{h}$ and $\mathfrak{g}$ the Lie algebras of H and G respectively, the differential of the canonical projection $\pi : G \to G/H$ at the identity identifies the orthogonal complement $\mathfrak{h}^\perp \subset \mathfrak{g}$ with the tangent space $T_o(G/H)$ at the origin $o = \{H\} \in G/H$. We put on $T_o(G/H)$ the induced metric (that is, the identification $\mathfrak{h}^\perp = T_o(G/H)$ becomes an isometry) and extend it by τ to a metric on G/H. This Riemannian metric on G/H is said to be *naturally reductive*. Obviously, $\pi : G \to G/H$ becomes a Riemannian submersion with totally geodesic fibres, in particular, π is harmonic [4].

Recall that a map $f : G/H \to S^n$ is equivariant if there exists a homomorphism $\rho : G \to SO(n+1)$ such that

$$f \circ \tau_g = \rho(g) \circ f, \qquad g \in G. \tag{3}$$

If f is full then ρ is uniquely determined; G acts, via ρ, on S^n and $\mathrm{im}(f) \subset S^n$ is an orbit of this action. Further, ρ is injective provided that f is an embedding and, in this case, (H) (that is the set of conjugacy classes of H in G) is the isotropy type of $\mathrm{im}(f)$ since $f : G/H \to S^n$ is an equivariant embedding.

Given a full equivariant embedding $f : G/H \to S^n$ (with the corresponding monomorphism $\rho : G \to SO(n+1)$) the pull-back bundle $\mathscr{F} = f^*(T(S^n))$ inherits a natural G-action by putting $g \cdot v_x = \rho(g)_* v_x \in \mathscr{F}_{\tau_g(x)}$, where $v_x \in \mathscr{F}_x$ $(= T_{f(x)}(S^n))$, $x \in G/H$, $g \in G$. Moreover, as G preserves the (induced) metric on $\mathscr{F}$ and leaves the subbundle $T(G/H)$ invariant, we have the G-invariant orthogonal decomposition

$$\mathscr{F} = \mathscr{N} \oplus T(G/H),$$

where $\mathscr{N}$ denotes the normal bundle of f. Denoting by $\mathscr{I}(\mathscr{F}) \subset C^\infty(\mathscr{F})$ the linear space of G-invariant sections of $\mathscr{F}$ we have the following.

THEOREM 1. *If $f : G/H \to S^n$ is a full equivariant harmonic embedding then $\mathscr{I}(\mathscr{F}) \subset K(f)$.*

Proof. Write $\rho : G \to SO(n+1)$ for the monomorphism satisfying (3); we first claim that

$$\Delta^G \rho = 2e(f) \cdot \rho\,, \tag{4}$$

where ρ is considered as a matrix-valued function on which the Laplacian Δ^G of G acts componentwise. Indeed, by the equivariant equation (3), as $\tau_g(o) = \pi(g)$, $g \in G$, we have

$$(\Delta^G \rho)(g) \cdot f(o) = \big(\Delta^G(f \circ \pi)\big)(g) = (\Delta^{G/H} f)\big(\pi(g)\big)$$
$$= 2e(f) \cdot f\big(\pi(g)\big) = 2e(f) \cdot \rho(g) \cdot f(o)\,.$$

The same argument works with o replaced by any point of G/H since the group metric is biinvariant. As f is full, (4) follows.

Now, putting $v \in \mathscr{I}(\mathscr{F})$, by Section 1, we have to show that

$$\Delta^{G/H} \hat{v} = 2e(f) \cdot \hat{v}\,.$$

Indeed, G-invariance of v and (4) implies that

$$(\Delta^{G/H} \hat{v})\big(\pi(g)\big) = \big(\Delta^G(\hat{v} \circ \pi)\big)(g) = (\Delta^G \rho)(g) \cdot \hat{v}(o)$$
$$= 2e(f) \cdot \rho(g) \cdot \hat{v}(o) = 2e(f) \cdot \hat{v}\big(\pi(g)\big)\,, \qquad g \in G\,;$$

and this completes the proof.

The principal property of infinitesimally rigid harmonic maps to be exploited subsequently is contained in the next result.

THEOREM 2. *Let $f : G/H \to S^n$ be a full equivariant infinitesimally rigid harmonic embedding with associated monomorphism $\rho : G \to SO(n+1)$. Then precomposition with f yields a linear isomorphism $\Phi : \mathfrak{z} \to \mathscr{I}(\mathscr{F})$, where $\mathfrak{z}$ denotes the Lie algebra of the centralizer of $\rho(G)$ in $SO(n+1)$. Further, Φ composed with the evaluation map $\varepsilon_o : C^\infty(\mathscr{F}) \to \mathscr{F}_o$ maps isomorphically onto* Fix $(H, \mathscr{F}_o)$, *where H acts on $\mathscr{F}_o$ via ρ.*

Proof. Put $X \in \mathfrak{z}$; now consider the one-parameter subgroup (ϕ_t) of isometries of S^n induced by X which is contained in the identity component Z_0 of the centralizer of $\rho(G) \subset SO(n+1)$. Here, for $g \in G$ and $x \in G/H$, we have

$$\big(g \cdot (X \circ f)\big)(x) = \rho(g)_*(X_{f(x)}) = \rho(g)_* \frac{d}{dt}\Big(\phi_t\big(f(x)\big)\Big)\Big|_{t=0}$$
$$= \frac{d}{dt}(\rho(g) \circ \phi_t)\big(f(x)\big)\big|_{t=0} = \frac{d}{dt}(\phi_t \circ \rho(g))\big(f(x)\big)\big|_{t=0}$$
$$= \frac{d}{dt}(\phi_t \circ f)\big(\tau_g(x)\big)\big|_{t=0} = \big((X \circ f) \circ \tau_g\big)(x)\,,$$

that is the vector field $X \circ f$ along f is G-invariant and on putting $\Phi(X) = X \circ f$, $X \in \mathfrak{z}$, we obtain a well-defined linear map $\Phi : \mathfrak{z} \to \mathscr{I}(\mathscr{F})$. As f is full and the connected components of the zero-set of a Killing vector field on S^n are totally geodesic submanifolds we conclude that Φ is injective. To prove that Φ is surjective let $v \in \mathscr{I}(\mathscr{F})$. By Theorem 1, $v \in K(f)$ $(= PK(f))$ and infinitesimal rigidity of f implies the existence of a Killing vector field $X \in so(n+1)$ with $v = X \circ f$. We claim that $X \in \mathfrak{z}$. Invariance of v implies that

$$(\rho(g)_* X) \circ f = (X \circ f) \circ \tau_g = (X \circ \rho(g)) \circ f, \qquad g \in G,$$

and so the Killing vector field $\rho(g)_* X - X \circ \rho(g)$ vanishes on $\mathrm{im}(f)$, that is we obtain

$$\rho(g)_* X = X \circ \rho(g), \qquad g \in G.$$

Denote by $(\phi_t) \subset SO(n+1)$ the one-parameter group of isometries induced by X; the last equation translates then as

$$\rho(g) \circ \phi_t = \phi_t \circ \rho(g), \qquad t \in \mathbb{R},$$

or equivalently, $(\phi_t) \subset Z_0$. Thus $X \in \mathfrak{z}$, and Φ is onto. The restriction $\varepsilon_0 | \mathscr{I}(\mathscr{F})$ maps $\mathscr{I}(\mathscr{F})$ isomorphically onto $\mathrm{Fix}(H, \mathscr{F}_0)$, since $v \in \mathscr{I}(\mathscr{F})$ is determined by its value v_o at the base point, and it is clearly necessary and sufficient to have $v_{\tau_h(o)} = h \cdot v_o$ for all $h \in H$.

3. *Infinitesimal flexibility*

The main result of this section is the following.

THEOREM 3. *Let $f : G/H \to S^n$, $n \geqslant 2$, be a full equivariant infinitesimally rigid harmonic embedding with associated monomorphism $\rho : G \to SO(n+1)$. Then, either f is onto or* $\mathrm{im}(f)$ *is a non-principal orbit of the action of G on S^n via ρ. In particular, in the latter case, if $H \subset G$ is connected (for example if G/H is simply connected) then* $\mathrm{im}(f)$ *is a singular orbit.*

Proof. Assuming that the orbit $\mathrm{im}(f) \subset S^n$ is principal we show that $\mathrm{im}(f) = S^n$. By a well-known property of principal orbits [4] H acts trivially on the normal space $\mathscr{N}_o$ (considered as a linear slice at $f(o) \in S^n$), that is $\mathscr{N}_o \subset \mathrm{Fix}(H, \mathscr{F}_o)$. We first claim that the closed subgroup $K = \rho(G) \cdot Z_0 \subset SO(n+1)$ acts on S^n transitively. Indeed, the tangent space $T_{f(o)}(K(f(o)))$ clearly contains $T_{f(o)}(\mathrm{im}(f))$ and, by Theorem 2, it contains $\mathscr{N}_o \subset \varepsilon(o)(\mathfrak{z})$. Thus $K(f(o))$ is open; since K is compact, the claim follows.

Next, we assert that all orbits of G on S^n have the same type. If $y, y' \in S^n$ then choose $k = \rho(g) \cdot z \in K$, $g \in G$, $z \in Z_0$, such that $y' = k(y)$. Then

$$G_{y'} = G_{\rho(g)(z(y))} = g \cdot G_{z(y)} \cdot g^{-1} = g \cdot G_y \cdot g^{-1},$$

or equivalently, $(G_y) = (G_{y'})$.

By Borel's classification of actions on S^n with one isotropy type (cf. [4, p. 196]), it follows that G is either transitive on S^n, that is we have $\mathrm{im}(f) = S^n$, or $G = S^3$ and G

acts freely on S^n. (Note that if $G = S^1$ the harmonic map f is either constant or maps onto a closed geodesic and hence fullness of f implies that $n \leqslant 1$, which is excluded.) Assuming that $G = S^3$, as G acts freely on S^n we have $H = \{1\}$, in particular,

$$\dim \mathfrak{z} = \dim \mathrm{Fix}(H, \mathscr{F}_o) = \dim \mathscr{F}_o = n.$$

By the proof of Theorem 2, Z_0 acts transitively on S^n since $T_{f(o)}(Z_0(f(o))) = \mathscr{F}_o$. For reasons of dimensions, Z_0 is actually diffeomorphic with S^n and hence $n = 3$. Thus $f : G \to S^3$ is onto and the proof is finished.

REMARK. If G/H is symmetric and $f : G/H \to S^n$ is an equivariant harmonic diffeomorphism then f is totally geodesic (cf. [**10**, proof of Corollary 1]). Further, if G/H is irreducible then the pull-back of the metric on S^n via f is G-invariant and so coincides (up to a constant multiple) with the one given on G/H, that is we obtain that $f : G/H \to S^n$ is homothetic. Note that any homothetic diffeomorphism $f : G/H \to S^n$ is infinitesimally rigid [**9**]. In particular, if G/H is irreducible symmetric then a harmonic diffeomorphism $f : G/H \to S^n$ with $e(f) = \text{const.}$ is infinitesimally rigid if and only if f is homothetic (compare with Corollary 2 in [**10**]). On the other hand, any conformal diffeomorphism of S^2 is infinitesimally rigid but non-isometric unless $e(f) = \text{const.}$ [**10**].

EXAMPLE 1. The standard Veronese surface $f : \mathbb{R}P^2 \to S^4$ is infinitesimally rigid [**6**]. Denoting by $\rho : SO(3) \to SO(5)$ the associated monomorphism, as $\mathbb{R}P^2$ is non-orientable, $\mathrm{im}(f) \subset S^n$ is a singular orbit [**4**, p. 188], in accordance with Theorem 3. Also, the isotropy subgroup O(2) is easily seen to act on $\mathscr{F}_o$ with trivial fixed point set.

In the case when $H = \{1\}$ Theorem 3 reduces to the following.

COROLLARY 1. *With the exception of homothetic diffeomorphisms* $f : G \to S^3$, *any full harmonic embedding* $f : G \to S^n$, $n \geqslant 2$, *with* $e(f) = \text{const.}$ *is infinitesimally non-rigid.*

For homogeneous hypersurfaces Theorem 3 can be sharpened as follows.

COROLLARY 2. *Let* G/H *be naturally reductive. Then any harmonic codimension-one embedding* $f : G/H \to S^n$, $n \geqslant 2$, *with* $e(f) = \text{const.}$, *is infinitesimally non-rigid.*

Proof. Assuming, on the contrary, that $f : G/H \to S^n$ is infinitesimally rigid, Theorem 3 implies that the orbit $\mathrm{im}(f) \subset S^n$ is exceptional. (Obviously, $\mathrm{im}(f) \subset S^n$ cannot be singular since otherwise G acts transitively on S^n.) As $\mathrm{im}(f)$ is non-principal we have $\mathrm{Fix}(H, \mathscr{N}_o) \neq \mathscr{N}_o$ and H acts on the line $\mathscr{N}_o$ with trivial fixed point set. Thus, according to the terminology of [**4**, p. 185], $\mathrm{im}(f)$ is special exceptional. As $n \geqslant 2$, we have $H_1(S^n; \mathbb{Z}_2) = 0$ which contradicts [**4**, 3.12. Theorem].

REMARK. Fullness of f in Corollary 2 is essential since, by [**9**], the canonical inclusion $f : SO(n)/SO(n-1) \to S^n$ is infinitesimally rigid.

EXAMPLE 2. The Clifford map $f : \mathbb{T}^2 \to S^3$ defined by

$$f(\phi, \psi) = (\cos\phi \cdot \cos\psi, \sin\phi \cdot \cos\psi, \cos\phi \cdot \sin\psi, \sin\phi \cdot \sin\psi), \qquad 0 \leqslant \phi, \psi < 2\pi,$$

is a full equivariant minimal hypersurface and so, by Corollary 1 or 2, infinitesimally non-rigid. Hence, by [**9**, Proposition 1], $\dim K(f) > \dim so(4) = 6$. In fact, $\dim K(f) = 7$ as the following argument shows.

Since $e(f) = 1$, to compute $\dim K(f)$ we have to determine the vector space of vector functions $\hat{v} : \mathbb{T}^2 \to \mathbb{R}^4$ satisfying

$$\Delta^{\mathbb{T}^2}\hat{v} = 2\hat{v} \tag{5}$$

with the linear constraint

$$\langle f, \hat{v} \rangle = 0. \tag{6}$$

By (5), the components of $\hat{v}$ are eigenfunctions of $\Delta^{\mathbb{T}^2}$ and hence [**1**] we have $\hat{v} = A \cdot f$, where A is a (4×4) matrix. Substituting this into $\langle f, \hat{v} \rangle$ we obtain a fourth-order homogeneous trigonometric polynomial whose coefficients, by (6), have to vanish. An easy computation shows that A has the form

$$A = \begin{bmatrix} 0 & \alpha & \beta & e_1 \\ -\alpha & 0 & e_2 & \gamma \\ -\beta & e_3 & 0 & \delta \\ e_4 & -\gamma & -\delta & 0 \end{bmatrix},$$

where $\alpha, \beta, \gamma, \delta, e_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, and $e_1 + e_2 + e_3 + e_4 = 0$. Thus, $\dim K(f) = 7$ and the claim follows.

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