Flexible harmonic maps into spheres

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1. INTRODUCTION AND PRELIMINARY RESULTS

In [15], to any harmonic map $f: M \to S^n$ [6] of a compact oriente Riemannian manifold M into the Euclidean n-sphere S^n there is associate a finite-dimensional vector space $K(f)$ consisting of all Jacobi fields v alo f [6] whose generalized divergence trace $\langle f_*, \nabla v\rangle$ ([11], p. 113) is zer where f_* is the differential of f; ∇ and $\langle\cdot,\cdot\rangle$ denote the induced connectic and metric of the "Riemannian-connected bundle $\mathscr{F} \otimes \Lambda_*(T_*(M)$ ($\mathscr{F}$ f $(T(S^n))$ resp.). Denoting by $PK(f) \subset K(f)$ the linear subspace of a projectable elements, we have $so(n+1)\circ f \subset PK(f)$ [14], where $so(n+1)$ considered as the Lie algebra of all infinitesimal isometries of S^n. The ma f is said to be *infinitesimally rigid* [15] if $so(n+1) \cdot f = PK(f)$ holds.

To introduce the corresponding local notion we note first that a vect field v along f is a *harmonic variation* of f (i.e. $f_t = \exp \cdot (tv)$: $M \to S^n$ harmonic for all $t \in \mathbb{R}$) if and only if $v \in K(f)$ with $\|v\|$ a constant [13].

The variation space $V(f)$ of f (i.e. the space consisting of all harmon variations of f) can then be considered as a subset of $K(f)$ with $V(f) = \mathbb{R}$ $V_0(f)$, where $V_0(f) = \{V \in K(f) | \; \|v\| = 1\}$. The map f is said to be *locall rigid* if for every projectable harmonic variation v there exists 1-parameter subgroup $(\varphi_t) \in SO(n+1)$ such that $f_t = \exp \cdot (tv) = \varphi_t \cdot f$ hol for all $t \in \mathbb{R}$. Obviously, infinitesimal and local rigidity is preserved und performing isometries of S^n.

The purpose of this paper is to study infinitesimal and local rigidity harmonic maps and to describe a variety of classical examples from th viewpoint. In Section 2 we first give a necessary and sufficient condition f the infinitesimal rigidity of harmonic maps $f: M \to S^n$.

As direct consequences we obtain that the inclusion $f: S^m \to S^n$, $m \leqslant$ the Veronese surface $f: S^2 \to S^4$ [2] and the Delaunay map $f: T^2 \to S^2$ [6] a infinitesimally rigid. Second, we prove that full infinitesimally rig harmonic embeddings $f: M \to S^n$ with energy density $e(f)$ a constant [6] a equivariant with respect to a faithful representation $\rho: i(M)_0 \to SO(n + 1$

where $i(M)_0$ denotes the identity component of the full isometry group of M (Theorem 1). Section 3 is devoted to the rigidity properties of the standard minimal immersions $f:S^m \to S^n$ defined by spherical harmonics of order s [4].

For $s = 2$ we show that f is infinitesimally rigid if and only if $m = 2$ (in which case the map $f:S^2 \to S^4$ is the Veronese surface). Moreover, for odd $m \geqslant 3$, the standard minimal immersions are non locally rigid (Theorem 2).

All manifolds, maps, bundles, etc., considered here will be of class C^∞. The report [6] is our general reference for the basic notions of the theory of harmonic maps and we adopt the sign conventions of [8].

2. INFINITESIMAL RIGIDITY AND EQUIVARIANCE OF HARMONIC MAPS

Let M be a compact oriented Riemannian manifold and consider a harmonic map $f:M \to S^n$. Denote Span (f) the intersection of all (closed) totally geodesic submanifolds of S^n which contaim the image of f. Then Span $(f) \subset S^n$ is a totally geodesic submanifold. The map f is said to be *full* if Span $(f) = S^n$. Our first result gives a simple criterion for the infinitesimal rigidity of f as follows:

Proposition 1 *For any harmonic map $f:M \to S^n$ with $r =$* dim Span (f) *we have*

$$\dim PK(f) \geqslant \frac{r(r+1)}{2} + (n-r)(r+1),$$

and equality holds if and only if f is infinitesimally rigid.

Proof By $so(n+1) \cdot f \subset PK(f)$ [14] the linear map Φ: $so(n+1) \to PK(f)$, $\Phi(X) = X \circ f$, $X \in so(n+1)$, is well defined. Thus dim $PK(f) \geqslant$ dim $so(n+1)$ $-$ dim ker Φ is valid. Moreover, f is infinitesimally rigid if and only if Φ is surjective, i.e. when equality holds. On the other hand, if $X \in$ ker Φ then X vanishes on Span (f) ([7], p.60) and so there is a linear isomorphism between the elements of ker Φ and the infinitesimal isometries defined on the cut locus of Span $(f) \subset S^n$. Hence ker $\Phi \cong so(n-r)$, which completes the proof. □

Example 1 According to a result of [15], dim $K(f) = m(m+1)/2 + (n-m)(m+1)$, where $f:S^m \to S^n$ is the inclusion. By Proposition 1 f is infinitesimally rigid.

Example 2 For any harmonic embedding $f:T^2 \to S^3$ with $e(f) = ½$ we have dim $K(f) = 7$ (>6) [14] and hence f is non-infinitesimally rigid.

Example 3 By [9], dim $K(f) = 10$ for the Veronese surface $f: S^2 \to S$
Again, by Proposition 1, the Veronese surface is infinitesimally rigid.

Example 4 Let α be the periodic solution of the pendulum equation

$$\ddot{x} + \sin x \cos x = 0$$

with initial values $\alpha(0) = 0$ and $\dot{\alpha}(0) = d$ $(0 < |d| < 1)$. Denoting by ω th period of α, parametrize the 2-torus T^2 with the Euclidean coordinates 0 : $\varphi < \omega$ and $0 \leqslant \psi < 2\pi$. Then the Delaunay map $f: T^2 \to S^2$ [6] defined b

$$f(\varphi, \psi) = (-\cos \psi \cos \alpha(\varphi), -\sin \psi \cos \alpha(\varphi), \sin \alpha(\varphi)),$$
$$0 \leqslant \varphi < \omega,\ 0 \leqslant \psi < 2\pi,$$

is harmonic with energy density $e(f) = \frac{1}{2}(\dot{\alpha}^2 + \cos^2 \alpha)$. In what follows w compute dim $K(f)$. Identifying vectors tangent to $S^2 \subset \mathbb{R}^3$ with thei translates at the origin, any vector field v along f gives rise to a map $\hat{v}: T^2 -$ $\mathbb{R}^3$ with $\langle f, \hat{v} \rangle = 0$. According to a result of [9], $v \in K(f)$ if and only if

$$(1) \qquad \Delta v_i = (\dot{\alpha}^2 + \cos^2\alpha) v_i, \qquad i = 1, 2, 3,$$

is valid, where $\hat{v} = (v_1, v_2, v_3)$ and Δ denotes the Laplacian on T^2. For fixe $i = 1, 2, 3$, the scalar v_i, being considered as a doubly periodic function o $\mathbb{R}^2$, has Fourier series expansion

$$v_i(\varphi, \psi) = p_0(\varphi) + \sum_{k=1}^{\infty} (p_k(\varphi)\cos(k\psi) + q_k(\varphi)\sin(k\varphi)),$$
$$0 \leqslant \varphi < \omega, \qquad 0 \leqslant \psi < 2\pi,$$

which splits (1) into the system

$$(2)_k \qquad \ddot{y}_k + (\dot{\alpha}^2 + \cos^2 \alpha - k^2) y_k = 0, \qquad k = 0, 1, 2, \ldots,$$

and p_k, q_k are solutions of $(2)_k$. On the other hand, by $\dot{\alpha}^2 + \sin^2\alpha = d^2$, we have $\dot{\alpha}^2 + \cos^2\alpha = d^2 - 1 + 2\cos^2\alpha < 2$, i.e. for $k \geqslant 2$ we have $\dot{\alpha}^2 + \cos^2\alpha$ $- k^2 < 0$ and hence periodicity of p_k and q_k imply that $p_k = q_k = 0$, $k \geqslant 2$, holds. Moreover, the function

$$\varphi \to \cos \alpha(\varphi) \int_0^{\varphi} \frac{dt}{\cos^2\alpha(t)}$$

is a non-periodic solution of $(2)_1$ and thus there exist $a_i, b_i \in \mathbb{R}$ such that

$$p_1 = a_i \cos \alpha \quad \text{and} \quad q_1 = b_i \cos \alpha$$

hold. Finally, $\sin \alpha$ being a periodic solution of $(2)_0$, we have

$$p_0 = c_i \sin \alpha + d_i y_0, \qquad c_i, d_i \in \mathbb{R}$$

where y_0 is the periodic solution of $(2)_0$ with initial values $y_0(0) = 1, \dot{y}_0 = 0$, and if it is not periodic we put $d_i = 0$. (Note that y_0 can be expressed in the singular integral form

$$\varphi \longrightarrow \sin \alpha(\varphi) \int_{\varphi}^{\omega/4} \frac{\mathrm{d}t}{\sin^2\alpha(t)}$$

Thus we obtain that

$$v_i(\varphi, \psi) = (a_i \cos \psi + b_i \sin \psi)\cos \alpha + c_i \sin \alpha + d_i y_0,$$
$$0 \leqslant \varphi < \omega, 0 \leqslant \psi < 2\pi,$$

holds for i=1, 2, 3. It remains only to satisfy the condition

$$\begin{aligned}(3)\quad 0 &= \langle f, \hat{v}\rangle \\ &= -\cos \psi \cos \alpha(\varphi)v_1(\varphi, \psi) \\ &\quad - \sin \psi \cos \alpha(\varphi)v_2(\varphi, \psi) \\ &\quad + \sin \alpha(\varphi)v_3(\varphi, \psi), \qquad 0 \leqslant \varphi < \omega, 0 \leqslant \psi < 2\pi.\end{aligned}$$

A straightforward computation, comparing the initial values of the various solutions, shows that (3) is valid if and only if $a_1 = b_2 = c_3 = d_1 = d_2 = d_3 = 0$, $a_2 + b_1 = a_3 - c_1 = b_3 - c_2 = 0$, i.e. putting $\rho = b_1$, $\sigma = c_1$ and $\tau = c_2$ we have

$$\begin{aligned}\hat{v}(\varphi, \psi) = (&\rho \sin \psi \cos \alpha(\varphi) + \sigma \sin \alpha(\varphi), \\ &-\rho \cos \psi \cos \alpha(\varphi) + \tau \sin \alpha(\varphi), \\ &(\alpha \cos \psi + \tau \sin \psi)\cos \alpha(\varphi)), \qquad 0 \leqslant \varphi < \omega, 0 \leqslant \psi < 2\pi;\end{aligned}$$

in particular, dim $K(f) = 3$. Applying Proposition 1 we obtain that the Delaunay map $f: T^2 \rightarrow S^2$ is infinitesimally rigid. Moreover, max rank $f = 2$ implies [15] that f is locally rigid as well.

In the rest of this section we show that certain infinitesimally rigid harmonic maps $f: M \rightarrow S^n$ are equivariant with respect to isometries. More precisely, we have the following:

Theorem 1 *Let $f: M \rightarrow S^n$ be a full infinitesimally rigid harmonic map with $e(f)$=const such that all elements of $K(f)$ are projectable. Then there exists a (unique) homomorphism $\rho: i(M)_0 \rightarrow SO(n + 1)$ of the identity component of the group of isometries of M into $SO(n + 1)$ such that f is ρ-equivariant.*

Remark The converse of Theorem 1 does not hold in general, as the following example shows:

Example 5 The harmonic embedding $f: T^2 \to S^3$ defined by

$$f(\varphi,\psi) = \left(\frac{1}{\sqrt{2}} \cos \varphi, \frac{1}{\sqrt{2}} \sin \varphi, \frac{1}{\sqrt{2}} \cos \psi, \frac{1}{\sqrt{2}} \sin \psi\right),$$

$$0 \leqslant \varphi, \psi < 2\pi$$

is equivariant with respect to the inclusion ρ: $T^2 \to SO(4)$ (as a maxima torus) given by

$$\rho(\varphi, \psi) = \operatorname{diag}\left(\begin{bmatrix}\cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi\end{bmatrix}, \begin{bmatrix}\cos \psi & -\sin \psi \\ \sin \psi & \cos \psi\end{bmatrix}\right),$$

$0 \leqslant \varphi, \psi < 2\pi$, though, by $e(f) = \frac{1}{2}$, Example 2 shows that f is no infinitesimally rigid.

In order to prove Theorem 1 first we state the following:

Lemma *Let f: $M \to S^n$ be a harmonic map with constant energy density. Then $f_*(X) \in K(f)$ holds for any infinitesimal isometry X of M.*

Proof Denote by $(\varphi_t) \subset i(M)_0$ the one-parameter group of isometries induced by X. Since $f_t = f \circ \varphi_t : M \to S^n$, $t \in \mathbb{R}$, is harmonic, $\partial f_t/\partial t\,|_{t=0} = f_*(X)$ is a Jacobi field along f [6], i.e. it remains only to prove that its generalized divergence vanishes. Fixing $x \in M - \text{Zero}(X)$, choose a local orthonormal frame $\{E^1, \ldots, E^m\}$ defined around x with $[X, E^i] = 0$. Then, using symmetry of the second fundamental form $\beta(f)$, we have

$$\begin{aligned}
\operatorname{trace} \langle f_*, \nabla(f_*(X))\rangle &= \sum_{i=1}^m \langle f_*(E^i), \nabla_{E_i}(f_*(X))\rangle \\
&= \sum_{i=1}^m \langle f_*(E^i), (\nabla_{E^i} f_*)X\rangle + \sum_{i=1}^m \langle f_*(E^i, f_*(\nabla_{E^i} X)\rangle \\
&= \sum_{i=1}^m \langle f_*(E^i), (\nabla_X f_*)E^i\rangle + \sum_{i=1}^m \langle f_*(E^i), f_*(\nabla_{E^i} X)\rangle \\
&= \sum_{i=1}^m \langle f_*(E^i), \nabla_X(f_*(E^i))\rangle - \sum_{i=1}^m \langle f_*(E^i), f_*(\nabla_X E^i)\rangle \\
&\quad + \sum_{i=1}^m \langle f_*(E^i), f_*(\nabla_{E^i} X)\rangle \\
&= X(e(f)) + \sum_{i=1}^m \langle f_*(E^i), f_*([E^i, X])\rangle = 0.
\end{aligned}$$

Since $\text{Zero}(X) \in M$ is nowhere dense, we obtain $\operatorname{trace} \langle f_*, \nabla(f_*(X))\rangle = 0$, which completes the proof. □

Remark Differentiating the equation $\Delta f_t = 2e(f_t)f_t$ by t at 0, where f_t is considered as a function taking its values in $\mathbb{R}^{n+1}$, and using a result of [9], a different proof of the lemma can be obtained.

Proof of Theorem 1 Let $f:M \to S^n$ be as in the theorem and put $g \in i(M)_0$. Identifying the elements of the Lie algebra of $i(M)_0$ with the infinitesimal isometries of M, compactness of $i(M)_0$ implies the existence of an infinitesimal isometry X of M such that $\exp X = g$; in particular, $\varphi_1 = g$, where (φ_t) is the one-parameter group of isometries induced by X. By the lemma, $f_*(X) \in K(f) = PK(f)$ and infinitesimal rigidity of f yields that $f_*(X) = Y \circ f$ holds for some $Y \in so(n+1)$. Thus the infinitesimal isometries X and Y are f-related [7] and it follows that $f \circ \varphi_t = \psi_t \circ f, f \in \mathbb{R}$, is valid, where $(\psi_t) \subset SO(n+1)$ is the one-parameter subgroup induced by Y. In particular, we obtain the existence of an isometry $\bar{g} \in SO(n+1)$ such that
$f \circ g = \bar{g} \circ f$ holds. Since f is full, $\bar{g}$ is uniquely determined. Putting $\bar{g} = \rho(g)$ we get a homomorphism ρ: $i(M)_0 \to SO(n+1)$ such that f is ρ-equivariant and the proof is finished.

Corollary 1 *Let M be symmetric and $f:M \to S^n$ a full infinitesimally rigid harmonic embedding with constant energy density. Then the second fundamental form $\beta(f))=\nabla f_*)$ of f is orthogonal to* im f, i.e. *we have $\langle\beta(f)(X, Y), f_*(Z)\rangle=0$ for all $X, Y, Z \in \mathfrak{X}(M)$.*

Proof Denote by $\omega \in \Theta^2 T^*(M)$ the pull-back of the metric tensor of S^n via f. By Theorem 1 there exists a monomorphism ρ: $i(M)_0 \to SO(n+1)$ such that f is ρ-equivariant. Thus, for $X, Y \in \mathfrak{X}(M)$ and $g \in i(M)_0$, we have $(g_*\omega)(X,Y) = \omega(g_*(X), g_*(Y)) = \langle f_*(g_*(X)), f_*(g_*(Y))\rangle = \langle\rho(g)_*(f_*(X)), \rho(g)_*(f_*(Y))\rangle = \langle f_*(X), f_*(Y)\rangle = \omega(X, Y)$, i.e. the symmetric 2-form ω is $i(M)_0$-invariant. Since M is symmetric it follows that ω is parallel ([**10**], p.174), i.e. for any $X, Y, Z \in \mathfrak{X}(M)$

$$(\nabla_X\omega)(Y, Z) = X(\omega(Y,Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) = 0$$

is valid, or equivalently

$$X\langle f_*(Y), f_*(Z)\rangle = \langle f_*(\nabla_X Y), f_*(Z)\rangle + \langle f_*(Y), f_*(\nabla_X Z)\rangle.$$

On the other hand, we have

$$X\langle f_*(Y), f_*(Z)\rangle = \langle\nabla_X(f_*(Y)), f_*(Z)\rangle + \langle f_*(Y), \nabla_X(f_*(Z))\rangle$$

and subtracting this equation from the above we obtain

$$0 = \langle\beta(f)(X, Y), f_*(Z)\rangle + \langle\beta(f)(X, Z), f_*(Y)\rangle,$$

where

$$\beta(f)(X, Y) = (\nabla_X f_*(Y)) - f_*(\nabla_X Y).$$

Hence, using the symmetry of $\beta(f)$ [6], we get

$$\begin{aligned}\langle \beta(f)(X, Y), f_*(Z)\rangle &= \langle \beta(f)(Y, X), f_*(Z)\rangle \\ &= -\langle \beta(f)(Y, Z), f_*(X)\rangle \\ &= -\langle \beta(f)(Z, Y), f_*(X)\rangle \\ &= \langle \beta(f)(Z, X), f_*(Y)\rangle \\ &= \langle \beta(f)(X, Z), f_*(Y)\rangle \\ &= -\langle \beta(f)(X, Y), f_*(Z)\rangle,\end{aligned}$$

which completes the proof. □

As a direct consequence we obtain the following characterization isometries of S^n:

Corollary 2 *A diffeomorphism $f: S^n \to S^n$ is isometric if and only if f is infinitesimally rigid harmonic map with constant energy density.*

Proof The identity id_{S^n}, and hence any isometry of S^n, is infinitesima rigid as it was established in [15]. Conversely, the hypotheses of Corolla 1 being satisfied, f is totally geodesic, from which we easily deduce that f isometric.

Example 6 Any conformal diffeomorphism $f: S^2 \to S^2$ is infinitesimal rigid. Indeed, it is just a reformulation of a result of Smith ([11], p 113–117). Clearly, $e(f) \not\equiv$ const unless f is isometric. (We note that $V(f) =$ since $\deg f = \pm 1$ [13]; in particular f is locally rigid.)

3. STANDARD MINIMAL IMMERSIONS

Consider the eigenspace $\mathcal{H}_{\lambda(s)}$ of the Laplacian $\Delta = \Delta^{S^m}$ of the Euclidea m-sphere S^m corresponding to the eigenvalue $\lambda(s) = s(s + m - 1)$, $s \in \mathbb{N}$ An element of $\mathcal{H}_{\lambda(s)}$ is the restriction (to S^m) of a homogeneous harmoni polynomial on $\mathbb{R}^{m+1}$ of degree s and so $\dim \mathcal{H}_{\lambda(s)} = n(s) + 1$, where

$$n(s) = (2s + m - 1)\,\frac{(s + m - 2)!}{s!(m - 1)!} - 1,$$

[2]. Integration over S^m defines an Euclidean scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_{\lambda(s)}$. Taking an orthonormal base $\{f^1, \ldots, f^{n(s)+1}\} \subset \mathcal{H}_{\lambda(s)}$ we have

$$\sum_{i=1}^{n(s)+1} (f^i)^2 = \text{const}$$

[4] and hence, by a suitable normalizing factor $N > 0$, the standard *minimal immersion* $f{:}S^m \to S^{n(s)}$ is defined (up to equivalence) by $f(x) = (Nf^1(x), \ldots, Nf^{n(s)+1}(x))$, $x \in S^m$. Then [2] f is full and homothetic such that

$$\|f_*(x)\|^2 = \frac{1}{k(s)}\,\|x\|^2$$

holds for $X \in \mathfrak{X}(S^m)$, where $k(s) = m/s(s + m - 1)$. Finally, for odd s, the map $f{:}S^m \to S^{n(s)}$ is an embedding and, for even s, it factors through the canonical projection $\pi{:}S^m \to \mathbb{R}P^m$, yielding an embedding $f{:}\mathbb{R}P^m \to S^{n(s)}$, [4].

Theorem 2 *If $s = 2$ then for the standard minimal immersion $f{:}S^m \to S^{n(2)}$ we have*

$$\dim PK(f) = \frac{1}{24}\,(5m^4 + 26m^3 + 19m^2 - 50m) - 1.$$

In particular, f is infinitesimally rigid if and only if $m = 2$ (i.e. $f{:}S^2 \to S^4$ is the Veronese surface).

Proof The second statement follows from the first by direct computation using Proposition 1. To prove the first, we note that an element of $\mathrm{H}_{\lambda(2)}$ is the restriction to S^m of a polynomial $p{:}\mathbb{R}^{m+1} \to \mathbb{R}$ which has the form

$$p = \sum_{k=1}^{m+1} a_k \varphi_k + 2\sum_{i<j} b_{ij}\varphi_{ij},$$

where $a_k, b_{ij} \in \mathbb{R}$ with

$$\sum_{k=1}^{m+1} a_k = 0$$

and

φ_k, φ_{ij} are scalars on S^m defined by $\varphi_k(x) = x_k^2$, $\varphi_{ij}(x) = x_i x_j$, $x = (x_1, \ldots, x_{m+1}) \in S^m$, $k = 1, \ldots, m + 1$, $1 \leqslant i < j \leqslant m + 1$ (cf. [2], p.176). The standard minimal immersion $f{:}S^m \to S^{n(2)}$ is then defined by the more explicit form

$$f(x_1, \ldots, x_{m+1}) = \frac{N}{I - J}\sum_{k=1}^{m+1}\left(x_k^2 - \frac{1}{m+1}\right)\varphi_k + \frac{2N}{J}\sum_{i<j} x_i x_j \varphi_{ij}, x \in S^m,$$

where $I = \|\varphi_k\|^2$, $J = \|\varphi_{ij}\|^2$ and $N > 0$ is a normalizing factor given by th condition $\|f\| = 1$. A vector field v along f determines a map $\hat{v}: S^m \to \mathcal{H}_{\lambda(2}$ given by translating the vectors of v to the origin of $\mathcal{H}_{\lambda(2)}$. Thus $\hat{v}$ has th form

$$\hat{v} = \sum_{k=1}^{m+1} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},$$

where a_k, b_{ij} are scalars on S^m with

$$\sum_{k=1}^{m+1} a_k = 0.$$

Moreover, by [9], $v \in K(f)$ is equivalent to $\Delta \hat{v} = 2(m+1)v$ since

$$2e(f) = \text{trace } \|f_*\|^2 = \frac{m}{k(2)} = 2(m+1).$$

Thus, orthogonality relations for φ_k and φ_{ij} ([2], p. 176) imply that a_k, b_{ij}, $k = 1, \ldots, m+1$, $1 \leqslant i < j \leqslant m+1$, belong to $\mathcal{H}_{\lambda(2)}$ and so we have

$$a_r = \sum_{k=1}^{m+1} a_k^r \varphi_k + 2 \sum_{i<j} b_{ij}^r \varphi_{ij}, \qquad r = 1, \ldots, m+1,$$

and

$$b_{pq} = \sum_{k=1}^{m+1} a_k^{pq} \varphi_k + 2 \sum_{i<j} b_{ij}^{pq} \varphi_{ij}, \qquad 1 \leqslant p < q \leqslant m+1,$$

where a_k^r, b_{ij}^r, a_k^{pq}, $b_{ij}^{pq} \in \mathbb{R}$ such that

$$(1) \quad \sum_{k=1}^{m+1} a_k^r = 0 \quad \text{and} \quad \sum_{k=1}^{m+1} a_k^{pq} = 0, \qquad r = 1, \ldots, m+1, 1 \leqslant p < q \leqslant m+1,$$

hold. The relation $\sum_{k=1}^{m+1} a_k = 0$ translates into

$$(2) \quad \sum_{r=1}^{m+1} a_k^r = 0 \quad \text{and} \quad \sum_{r=1}^{m+1} b_{ij}^r = 0.$$

(In particular, $PK(f) = K(f)$.) Finally, as in the proof of Theorem 2 in [9], an easy computation shows that $\langle f, \hat{v} \rangle = 0$ is satisfied if and only if the following relations hold:

(3) $a_k^k = 0$, $k = 1, \ldots, m+1$;

(4) $a_j^i + a_i^j + 8b_{ij}^{ij} = 0$, $1 \leqslant i < j \leqslant m+1$;

(5) $b^{ij} + 2a_i^{ij} = 0$, $1 \leqslant i < j \leqslant m+1$;

(6) $b_{ij}^i + 2a_j^{ij} = 0$, $1 \leqslant i<j \leqslant m+1$;

(7) $b_{ij}^r + 2a_r^{ij} + 4b_{ri}^j + 4b_{rj}^i = 0$, $1 \leqslant r < i < j \leqslant m+1$;

(8) $b^r_{ij} + 2a^{ij}_r + 4b^{ir}_{rj} + 4b^{rj}_{ir} = 0$, $1 \leqslant i < r < j \leqslant m + 1$;

(9) $b^r_{ij} + 2a^{ij}_r + 4b^{jr}_{ir} + 4b^{ir}_{jr} = 0$, $1 \leqslant i < j \leqslant m + 1$;

(10) $b^{pq}_{ij} + b^{jq}_{ip} + b^{jp}_{iq} + b^{iq}_{jp} + b^{ip}_{jq} + b^{ij}_{pq} = 0$
for all $1 \leqslant i < j < p < q \leqslant m + 1$.

Our task is to compute $\dim K(f)$, i.e. the dimension of the linear space consisting of all solutions of the linear system (1)–(10). In order to solve the system (1)–(10) we first choose the coefficients a^r_k, k, $r = 1, \ldots, m + 1$, satisfying the corresponding relations in (1), (2) and (3). These span a linear space of dimension $(m + 1)^2 - ((m + 1) + m + 1 + m) = m^2 - m - 1$. The second relation of (2) yields that the linear space spanned by the coefficients b^r_{ij}, $r = 1, \ldots, m + 1$, $1 \leqslant i < j \leqslant m + 1$, has dimension $\binom{m+1}{2}m$. By relations (5)–(6) the coefficients a^{ij}_i, a^{ij}_j, $1 \leqslant i < j \leqslant m + 1$, depend on a^r_k. Hence, by the second relation of (1), each row

$$a^{ij}_1, a^{ij}_2, \ldots, a^{ij}_{m+1}, \qquad 1 \leqslant i < \mathrm{j} \leqslant \mathrm{m} + 1,$$

contains exactly three dependent elements, i.e. the dimension of the linear space of independent coefficients a^{ij}_r, $r = 1, \ldots, m + 1$, $1 \leqslant i < j \leqslant m + 1$, is $(m - 2)\binom{m+1}{2}$. Now we have to determine the linear space of coefficients b^{pq}_{ij}, $1 \leqslant i < j \leqslant m + 1$, $1 \leqslant p < q \leqslant m + 1$, which are independent from $\{a^r_k, b^r_{ij}, a^{ij}_r \,|\, k, r = 1, \ldots, m + 1, 1 \leqslant i < j \leqslant m + 1\}$. By relation (4) the elements b^{ij}_{ij}, $1 \leqslant i < j \leqslant m + 1$, are dependent. The linear space of independent coefficients b^{pq}_{ij} such that i, j, p, q are not mutually distinct, by relations (7), (8) and (9), is spanned by b^{rj}_{ri}, $1 \leqslant r < i < j \leqslant m + 1$, b^{ir}_{rj}, $1 \leqslant i < r < j \leqslant m + 1$, and b^{jr}_{ir}, $1 \leqslant i < j < r \leqslant m + 1$, i.e. its dimension is $3\binom{m+1}{3}$. Finally, the only relations between the coefficients b^{pq}_{ij} for which i, j, p, q are mutually different are in (10). Each equation in (10) contains exactly one coefficient b^{pq}_{ij} with $i < j < p < q$ and so the dimension of the corresponding linear space is $5\binom{m+1}{4}$. Thus we have

$$\dim K(f) = m^2 - m - 1 + \binom{m+1}{2}m + (m - 2)\binom{m+1}{2} + 3\binom{m+1}{3} + 5\binom{m+1}{4}.$$

and a straightforward computation completes the proof. □

Next we turn to the local rigidity of the standard minimal imersions. As has already been established in [9], the Veronese surface $f{:}S^2 \to S^4$ has trivial variation space, in particular, it is locally rigid. In contrast to this, for $m \geqslant 3$, we have the following:

Theorem 3 *If $m \geqslant 3$ is odd, then the standard minimal immersion $f{:}S^m \to S^{n(s)}$ is non-locally rigid.*

Proof First we show that local rigidity of $f:S^m \to S^{n(s)}$ implies that j totally geodesic, i.e it maps geodesics of S^m onto geodesics of $S^{m(s)}$ linea [6]. Let $\gamma:\mathbb{R}\to S^m$ be a geodesic with initial vector $X_{x_0} \in T_{x_0}(S^m)$, $x_0 = \gamma($ Because m is odd we can extend the vector X_{x_0} to an infinitesimal isomet $X \in so(m+1)$ with $\|X\| = \text{const}$. Then, denoting by $(\varphi_t) \subset SO(m+1)$ t one-parameter group of isometries induced by X, the integral curves t $\varphi_t(x)$, $x \in S^m$, are geodesics, in particular the vector field $v = f_*(X)$ alon is projectable. On the other hand, by the lemma, $v \in K(f)$ and, since

$$\|v\| = \|f_*(x)\| = \frac{1}{\sqrt{k(s)}}\,\|x\| = \text{const},$$

we get $v \in V(f)$. Local rigidity of f implies the existence of a one-paramet group $(\varphi_t) \subset SO(n(s)+1)$ of isometries such that $f_t = \exp \circ (tv) = \psi_t \circ$ holds for all $t \in \mathbb{R}$. Thus, if Y denotes the infinitesimal isometry induced (ψ_t) then $f_*(X) = Y \circ f$, i.e. the vector fields X and Y are f-related. follows that $f \circ \varphi_t = \psi_t \circ f$ is valid for all $t \in \mathbb{R}$, i.e. the geodesic $t \to \gamma(t)$ $\varphi_t(x_0)$ is mapped under f to the geodesic $t \to \exp(tv_{x_0})$ since $f(\varphi_t(x_0))$ $\psi_t(f(x_0)) = \exp(tv_{x_0})$, $t \in \mathbb{R}$, and hence f is totally geodesic. Clearly, th image of f is then a totally geodesic submanifold of $S^{n(s)}$ which contradic the fullness of f. Thus the theorem is proved. □

The only property of the domain used in the proof of Theorem 3 is th extendability of a given vector X_{x_0} to an infinitesimal isometry X with $\|X$ $=$ const. Thus, by the same argument, we get the following:

Corollary 3 *Let M be either an odd Euclidean sphere or a compact Li group with biinvariant metric. Then any full harmonic homotheti embedding $f:M \to S^n$, with* dim $M < n$, *is non-locally rigid.*

In particular, the harmonic embedding $f:T^2 \to S^3$ of Example 2 i non-locally rigid.

Remark According to Calabi [3] and Do carmo and Wallach [4] ful isometric minimal immersions of the sphere $S^m_{k(s)}$ of constant sectiona curvature $k(s)$ into $S^{n(s)}$ are rigid, provided that $s \leqslant 3$, i.e. for any two sucl maps $f', f'': S^m_{k(s)} \to S^{n(s)}$ there exists an isometry $\varphi \in O(n(s)+1)$ with $\varphi \circ f$ $= f'$. On the other hand, non-local rigidity of the standard minima immersion $f:S^m \to S^{n(s)}$,. for odd m and $s \leqslant 3$, and the rigidity theoren above imply the existence of a harmonic variation $v \in V(f)$ along which the deformed harmonic maps $f_t = \exp \circ (tv)$ will not be in general homothetic Thus we have examples of homothetic minimal immersions which are rigic among homothetic minimal immersions but non-rigid among harmonic maps.

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