Flexible harmonic maps into spheres

Gábor Tóth
Mathematics Institute, Hungarian Academy of Sciences, Budapest

1. INTRODUCTION AND PRELIMINARY RESULTS

In [15], to any harmonic map \( f: M \to S^n \) [6] of a compact orientable Riemannian manifold \( M \) into the Euclidean \( n \)-sphere \( S^n \) there is associate a finite-dimensional vector space \( K(f) \) consisting of all Jacobi fields \( \nu \) along \( f \) [6] whose generalized divergence trace \( \langle f^*, \nabla \nu \rangle \) ([11], p. 113) is zero, where \( f^* \) is the differential of \( f \), \( \nabla \) and \( \langle \cdot, \cdot \rangle \) denote the induced connection and metric of the "Riemannian-connected bundle \( \mathcal{T} \otimes \Lambda^* (T^* (M)) \) (\( \mathcal{T} \) \( T(S^n) \)) resp.). Denoting by \( PK(f) \subset K(f) \) the linear subspace of the projectable elements, we have \( so(n+1)_{cl} \subset PK(f) \) [14], where \( so(n+1) \) is considered as the Lie algebra of all infinitesimal isometries of \( S^n \). The map \( f \) is said to be **infinitesimally rigid** [15] if \( so(n+1) \cdot f = PK(f) \) holds.

To introduce the corresponding local notion we note first that a vector field \( \nu \) along \( f \) is a **harmonic variation** of \( f \) (i.e. \( f_t = \exp \cdot (tv) : M \to S^n \) harmonic for all \( t \in \mathbb{R} \)) if and only if \( \nu \in K(f) \) with \( ||\nu|| \) a constant [13].

The variation space \( V(f) \) of \( f \) (i.e. the space consisting of all harmonic variations of \( f \)) can then be considered as a subset of \( K(f) \) with \( V(f) = \{ V \in K(f) | || \nu || = 1 \} \). The map \( f \) is said to be **local rigid** if for every projectable harmonic variation \( \nu \) there exists 1-parameter subgroup \( (\varphi_t) \in SO(n+1) \) such that \( f_t = \exp \cdot (tv) = \varphi_t \cdot f \) holomorphically for all \( t \in \mathbb{R} \). Obviously, infinitesimal and local rigidity is preserved under performing isometries of \( S^n \).

The purpose of this paper is to study infinitesimal and local rigidity of harmonic maps and to describe a variety of classical examples from this viewpoint. In Section 2 we first give a necessary and sufficient condition for the infinitesimal rigidity of harmonic maps \( f: M \to S^n \).

As direct consequences we obtain that the inclusion \( f: S^m \to S^n \), \( m \leq \) the Veronese surface \( f: S^2 \to S^4 \) [2] and the Delaunay map \( f: T^2 \to S^2 \) [6] a infinitesimally rigid. Second, we prove that full infinitesimally rigid harmonic embeddings \( f: M \to S^n \) with energy density \( e(f) \) a constant [6] a equivariant with respect to a faithful representation \( \rho: i(M)_0 \to SO(n+1) \).
where \( i(M)_0 \) denotes the identity component of the full isometry group of \( M \) (Theorem 1). Section 3 is devoted to the rigidity properties of the standard minimal immersions \( f:S^m \to S^n \) defined by spherical harmonics of order \( s \) \([4]\).

For \( s = 2 \) we show that \( f \) is infinitesimally rigid if and only if \( m = 2 \) (in which case the map \( f:S^2 \to S^4 \) is the Veronese surface). Moreover, for odd \( m \geq 3 \), the standard minimal immersions are non locally rigid (Theorem 2).

All manifolds, maps, bundles, etc., considered here will be of class \( C^\infty \). The report \([6]\) is our general reference for the basic notions of the theory of harmonic maps and we adopt the sign conventions of \([8]\).

2. INFINITESIMAL RIGIDITY AND EQUIVARIANCE OF HARMONIC MAPS

Let \( M \) be a compact oriented Riemannian manifold and consider a harmonic map \( f:M \to S^n \). Denote \( \text{Span} (f) \) the intersection of all (closed) totally geodesic submanifolds of \( S^n \) which contain the image of \( f \). Then \( \text{Span} (f) \subset S^n \) is a totally geodesic submanifold. The map \( f \) is said to be full if \( \text{Span} (f) = S^n \). Our first result gives a simple criterion for the infinitesimal rigidity of \( f \) as follows:

**Proposition 1** For any harmonic map \( f:M \to S^n \) with \( r = \dim \text{Span} (f) \) we have

\[
\dim PK(f) \geq \frac{r(r + 1)}{2} + (n - r)(r + 1),
\]

and equality holds if and only if \( f \) is infinitesimally rigid.

**Proof** By \( so(n + 1) \cdot f \subset PK(f) \) \([14]\) the linear map \( \Phi: so(n + 1) \to PK(f), \Phi(X) = X \circ f, X \in so(n+1) \), is well defined. Thus \( \dim PK(f) \geq \dim so(n+1) - \dim \ker \Phi \) is valid. Moreover, \( f \) is infinitesimally rigid if and only if \( \Phi \) is surjective, i.e. when equality holds. On the other hand, if \( X \in \ker \Phi \) then \( X \) vanishes on \( \text{Span} (f) \) \([7], p.60\) and so there is a linear isomorphism between the elements of \( \ker \Phi \) and the infinitesimal isometries defined on the cut locus of \( \text{Span} (f) \subset S^n \). Hence \( \ker \Phi \equiv so(n-r) \), which completes the proof. \( \square \)

**Example 1** According to a result of \([15]\), \( \dim K(f) = m(m + 1)/2 + (n - m)(m + 1) \), where \( f:S^m \to S^n \) is the inclusion. By Proposition 1 \( f \) is infinitesimally rigid.

**Example 2** For any harmonic embedding \( f:T^2 \to S^3 \) with \( e(f) = ½ \) we have \( \dim K(f) = 7 \) \((>6)\) \([14]\) and hence \( f \) is non-infinitesimally rigid.
Example 3  By [9], dim $K(f) = 10$ for the Veronese surface $f: S^2 \to S$
Again, by Proposition 1, the Veronese surface is infinitesimally rigid.

Example 4  Let $\alpha$ be the periodic solution of the pendulum equation
$$\ddot{x} + \sin x \cos x = 0$$
with initial values $\alpha(0) = 0$ and $\dot{\alpha}(0) = d$ $(0 < |d| < 1)$. Denoting by $\omega$ the period of $\alpha$, parametrize the 2-torus $T^2$ with the Euclidean coordinates $0 < \varphi < \omega$ and $0 \leq \psi < 2\pi$. Then the Delaunay map $f: T^2 \to S^2$ [6] defined by
$$f(\varphi, \psi) = (-\cos \psi \cos \alpha(\varphi), -\sin \psi \cos \alpha(\varphi), \sin \alpha(\varphi), 0 \leq \varphi < \omega, 0 \leq \psi < 2\pi,$$
is harmonic with energy density $e(f) = \frac{1}{2}(\dot{\alpha}^2 + \cos^2 \alpha)$. In what follows we compute $\dim K(f)$. Identifying vectors tangent to $S^2 \subset \mathbb{R}^3$ with their translates at the origin, any vector field $\nu$ along $f$ gives rise to a map $\nu: T^2 \to \mathbb{R}^3$ with $(f, \nu) = 0$. According to a result of [9], $\nu \in K(f)$ if and only if
$$\Delta \nu_i = (\dot{\alpha}^2 + \cos^2 \alpha)\nu_i, \quad i = 1, 2, 3,$$
is valid, where $\nu = (\nu_1, \nu_2, \nu_3)$ and $\Delta$ denotes the Laplacian on $T^2$. For fixed $i = 1, 2, 3$, the scalar $\nu_i$, being considered as a doubly periodic function of $\mathbb{R}^2$, has Fourier series expansion
$$\nu_i(\varphi, \psi) = p_0(\varphi) + \sum_{k=1}^{\infty} (p_k(\varphi)\cos(k\psi) + q_k(\varphi)\sin(k\varphi)), \quad 0 \leq \varphi < \omega, \quad 0 \leq \psi < 2\pi,$$
which splits (1) into the system
$$(2k) \quad \ddot{y}_k + (\dot{\alpha}^2 + \cos^2 \alpha - k^2)y_k = 0, \quad k = 0, 1, 2, \ldots,$$
and $p_k, q_k$ are solutions of $(2k)$. On the other hand, by $\dot{\alpha}^2 + \sin^2 \alpha = d^2$, we have $\dot{\alpha}^2 + \cos^2 \alpha = d^2 - 1 + 2\cos^2 \alpha < 2$, i.e. for $k \geq 2$ we have $\dot{\alpha}^2 + \cos^2 \alpha - k^2 < 0$ and hence periodicity of $p_k$ and $q_k$ imply that $p_k = q_k = 0$, $k \geq 2$, holds. Moreover, the function
$$\varphi \to \cos \alpha(\varphi) \int_0^\varphi \frac{dt}{\cos^2 \alpha(t)}$$
is a non-periodic solution of (2), and thus there exist $a_i, b_i \in \mathbb{R}$ such that
$$p_1 = a_i \cos \alpha \quad \text{and} \quad q_1 = b_i \cos \alpha$$
hold. Finally, $\sin \alpha$ being a periodic solution of $(2)_0$, we have
$$p_0 = c_i \sin \alpha + d_i y_0, \quad c_i, d_i \in \mathbb{R}$$
where $y_0$ is the periodic solution of (2) with initial values $y_0(0) = 1, \dot{y}_0 = 0$, and if it is not periodic we put $d_i = 0$. (Note that $y_0$ can be expressed in the singular integral form

$$\varphi \rightarrow \sin \alpha(\varphi) \int_{\varphi}^{\omega} \frac{dt}{\sin^2 \alpha(t)}$$

Thus we obtain that

$$v_i(\varphi, \psi) = (a_i \cos \psi + b_i \sin \psi) \cos \alpha + c_i \sin \alpha + d_i y_0,$$

$$0 \leq \varphi < \omega, 0 \leq \psi < 2\pi,$$

holds for $i = 1, 2, 3$. It remains only to satisfy the condition

$$(3) \quad 0 = \langle f, \dot{v} \rangle$$

$$= -\cos \psi \cos \alpha(\varphi)v_1(\varphi, \psi)$$

$$- \sin \psi \cos \alpha(\varphi)v_2(\varphi, \psi)$$

$$+ \sin \alpha(\varphi)v_3(\varphi, \psi), \quad 0 \leq \varphi < \omega, 0 \leq \psi < 2\pi.$$ 

A straightforward computation, comparing the initial values of the various solutions, shows that (3) is valid if and only if $a_1 = b_2 = c_3 = d_1 = d_2 = d_3 = 0, a_2 + b_1 = a_3 - c_1 = b_3 - c_2 = 0$, i.e. putting $\rho = b_1, \sigma = c_1$ and $\tau = c_2$ we have

$$\dot{v}(\varphi, \psi) = (\rho \sin \psi \cos \alpha(\varphi) + \sigma \sin \alpha(\varphi),$$

$$-\rho \cos \psi \cos \alpha(\varphi) + \tau \sin \alpha(\varphi),$$

$$(\alpha \cos \psi + \tau \sin \psi) \cos \alpha(\varphi)), \quad 0 \leq \varphi < \omega, 0 \leq \psi < 2\pi;$$

in particular, dim $K(f) = 3$. Applying Proposition 1 we obtain that the Delaunay map $f: \mathbb{T}^2 \rightarrow S^2$ is infinitesimally rigid. Moreover, max rank $f = 2$ implies [15] that $f$ is locally rigid as well.

In the rest of this section we show that certain infinitesimally rigid harmonic maps $f: M \rightarrow S^n$ are equivariant with respect to isometries. More precisely, we have the following:

**Theorem 1** Let $f: M \rightarrow S^n$ be a full infinitesimally rigid harmonic map with $e(f) = \text{const}$ such that all elements of $K(f)$ are projectable. Then there exists a (unique) homomorphism $\rho: i(M)_0 \rightarrow SO(n + 1)$ of the identity component of the group of isometries of $M$ into $SO(n + 1)$ such that $f$ is $\rho$-equivariant.

**Remark** The converse of Theorem 1 does not hold in general, as the following example shows:
Example 5  The harmonic embedding \( f: T^2 \to S^3 \) defined by

\[
f(\varphi, \psi) = \left( \frac{1}{\sqrt{2}} \cos \varphi, \frac{1}{\sqrt{2}} \sin \varphi, \frac{1}{\sqrt{2}} \cos \psi, \frac{1}{\sqrt{2}} \sin \psi \right),
\]

\( 0 \leq \varphi, \psi < 2\pi \)

is equivariant with respect to the inclusion \( \rho: T^2 \to SO(4) \) (as a maximal torus) given by

\[
\rho(\varphi, \psi) = \text{diag} \left( \begin{bmatrix} \cos \varphi - \sin \varphi \\ \sin \varphi \cos \varphi \end{bmatrix}, \begin{bmatrix} \cos \psi - \sin \psi \\ \sin \psi \cos \psi \end{bmatrix} \right),
\]

\( 0 \leq \varphi, \psi < 2\pi \), though, by \( e(f) = \frac{1}{2} \), Example 2 shows that \( f \) is no infinitesimally rigid.

In order to prove Theorem 1 first we state the following:

Lemma  Let \( f: M \to S^n \) be a harmonic map with constant energy density. Then \( f_*(X) \in K(f) \) holds for any infinitesimal isometry \( X \) of \( M \).

Proof  Denote by \( (\varphi_t) \subset i(M)_0 \) the one-parameter group of isometries induced by \( X \). Since \( f_t = f \circ \varphi_t: M \to S^n, t \in \mathbb{R} \), is harmonic, \( \frac{d f_t}{dt} \big|_{t=0} = f_*(X) \) is a Jacobi field along \( f \) [6], i.e. it remains only to prove that its generalized divergence vanishes. Fixing \( x \in M - \text{Zero}(X) \), choose a local orthonormal frame \( \{ E^1, \ldots, E^n \} \) defined around \( x \) with \( [X, E^i] = 0 \). Then, using symmetry of the second fundamental form \( \beta(f) \), we have

\[
\text{trace} (f_*, \nabla (f_*(X))) = \sum_{i=1}^m (f_*(E^i), \nabla_{E^i} (f_*(X))) \\
= \sum_{i=1}^m (f_*(E^i), (\nabla_{E^i} f_*) X) + \sum_{i=1}^m (f_*(E^i), f_*(\nabla_{E^i} X)) \\
= \sum_{i=1}^m (f_*(E^i), (\nabla_X f_*) E^i) + \sum_{i=1}^m (f_*(E^i), f_*(\nabla_{E^i} X)) \\
= \sum_{i=1}^m (f_*(E^i), \nabla_X (f_*(E^i))) - \sum_{i=1}^m (f_*(E^i), f_*(\nabla_{E^i} X)) \\
+ \sum_{i=1}^m (f_*(E^i), f_*(\nabla_{E^i} X)) \\
= X(e(f)) + \sum_{i=1}^m (f_*(E^i), f_*(E^i)) = 0.
\]

Since \( \text{Zero}(X) \in M \) is nowhere dense, we obtain \( \text{trace} (f_*, \nabla (f_*(X))) = 0 \), which completes the proof. □

Remark  Differentiating the equation \( \Delta f_t = 2e(f_t)f \), by \( t \) at \( t = 0 \), where \( f_t \) is
considered as a function taking its values in $\mathbb{R}^{n+1}$, and using a result of [9], a different proof of the lemma can be obtained.

**Proof of Theorem 1** Let $f: M \to S^n$ be as in the theorem and put $g \in i(M)_0$. Identifying the elements of the Lie algebra of $i(M)_0$ with the infinitesimal isometries of $M$, compactness of $i(M)_0$ implies the existence of an infinitesimal isometry $X$ of $M$ such that $\exp X = g$; in particular, $\varphi_1 = g$, where $(\varphi_t)$ is the one-parameter group of isometries induced by $X$. By the lemma, $f^*(X) \in K(f) = PK(f)$ and infinitesimal rigidity of $f$ yields that $f^*(X) = Y \circ f$ holds for some $Y \in so(n + 1)$. Thus the infinitesimal isometries $X$ and $Y$ are $f$-related [7] and it follows that $f \circ \varphi_t = \psi_t \circ f$, $f \in \mathbb{R}$, is valid, where $(\psi_t) \subset SO(n + 1)$ is the one-parameter subgroup induced by $Y$. In particular, we obtain the existence of an isometry $\tilde{g} \in SO(n + 1)$ such that $f \circ g = \tilde{g} \circ f$ holds. Since $f$ is full, $\tilde{g}$ is uniquely determined. Putting $\tilde{g} = \rho(g)$ we get a homomorphism $\rho: i(M)_0 \to SO(n + 1)$ such that $f$ is $\rho$-equivariant and the proof is finished.

**Corollary 1** Let $M$ be symmetric and $f: M \to S^n$ a full infinitesimally rigid harmonic embedding with constant energy density. Then the second fundamental form $\beta(f) = \nabla f^*$ of $f$ is orthogonal to $\text{im} f$, i.e. we have $\langle \beta(f)(X, Y), f^*(Z) \rangle = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$.

**Proof** Denote by $\omega \in \Theta^2 T^* M$ the pull-back of the metric tensor of $S^n$ via $f$. By Theorem 1 there exists a monomorphism $\rho: i(M)_0 \to SO(n + 1)$ such that $f$ is $\rho$-equivariant. Thus, for $X, Y \in \mathfrak{X}(M)$ and $g \in i(M)_0$, we have $(g^*\omega)(X, Y) = \omega((g^*X), g^*(Y)) = (f^*(g^*(X)), f^*(g^*(Y))) = (\rho(g)^*(f^*(X)), \rho(g)^*(f^*(Y))) = (f^*(X), f^*(Y)) = \omega(X, Y)$, i.e. the symmetric 2-form $\omega$ is $i(M)_0$-invariant. Since $M$ is symmetric it follows that $\omega$ is parallel ([10], p.174), i.e. for any $X, Y, Z \in \mathfrak{X}(M)$

$$(\nabla_X \omega)(Y, Z) = X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) = 0$$

is valid, or equivalently

$$X(f^*(Y), f^*(Z)) = (f^*(\nabla_X Y), f^*(Z)) + (f^*(Y), f^*(\nabla_X Z)).$$

On the other hand, we have

$$X(f^*(Y), f^*(Z)) = (\nabla_X f^*(Y), f^*(Z)) + (f^*(Y), \nabla_X f^*(Z))$$

and subtracting this equation from the above we obtain

$$0 = \langle \beta(f)(X, Y), f^*(Z) \rangle + \langle \beta(f)(X, Z), f^*(Y) \rangle,$$

where

$$\beta(f)(X, Y) = (\nabla_X f^*(Y)) - f^*(\nabla_X Y).$$
Hence, using the symmetry of $\beta(f)$ [6], we get
\[(\beta(f)(X, Y), f^*(Z)) = (\beta(f)(Y, X), f^*(Z))
= -(\beta(f)(Y, Z), f^*(X))
= -(\beta(f)(Z, Y), f^*(X))
= (\beta(f)(Z, X), f^*(Y))
= (\beta(f)(X, Z), f^*(Y))
= -(\beta(f)(X, Y), f^*(Z)),\]
which completes the proof. \(\square\)

As a direct consequence we obtain the following characterization isometries of $S^n$:

**Corollary 2**  A diffeomorphism $f:S^n \to S^n$ is isometric if and only if $f$ is infinitesimally rigid harmonic map with constant energy density.

**Proof**  The identity $id_{S^n}$ and hence any isometry of $S^n$, is infinitesimally rigid as it was established in [15]. Conversely, the hypotheses of Corollary 1 being satisfied, $f$ is totally geodesic, from which we easily deduce that $f$ is isometric.

**Example 6**  Any conformal diffeomorphism $f:S^2 \to S^2$ is infinitesimal rigid. Indeed, it is just a reformulation of a result of Smith ([11], p. 113–117). Clearly, $e(f) \not= \text{const}$ unless $f$ is isometric. (We note that $V(f) = \text{deg} f = \pm 1$ [13]; in particular $f$ is locally rigid.)

### 3. STANDARD MINIMAL IMMERSIONS

Consider the eigenspace $\mathcal{H}_{\lambda(s)}$ of the Laplacian $\Delta = \Delta S^n$ of the Euclidean $m$-sphere $S^m$ corresponding to the eigenvalue $\lambda(s) = s(s + m - 1)$, $s \in \mathbb{N}$. An element of $\mathcal{H}_{\lambda(s)}$ is the restriction (to $S^m$) of a homogeneous harmonic polynomial on $\mathbb{R}^{m+1}$ of degree $s$ and so $\text{dim} \mathcal{H}_{\lambda(s)} = n(s) + 1$, where

\[n(s) = (2s + m - 1) \frac{(s + m - 2)!}{s!(m - 1)!} - 1,\]

[2]. Integration over $S^m$ defines an Euclidean scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_{\lambda(s)}$. Taking an orthonormal base $\{f^1, ..., f^{n(s)} + 1\} \subset \mathcal{H}_{\lambda(s)}$ we have

\[\sum_{i=1}^{n(s)+1} (f^i)^2 = \text{const}\]
and hence, by a suitable normalizing factor \( N > 0 \), the standard minimal immersion \( f:S^m \to S^{n(s)} \) is defined (up to equivalence) by \( f(x) = (Nd^1(x), \ldots, Nf^{n(s)} + 1(x)) \), \( x \in S^m \). Then \([2]\) \( f \) is full and homothetic such that

\[
\|f_\ast(x)\|^2 = \frac{1}{k(s)} \|x\|^2
\]

holds for \( X \in X(S^m) \), where \( k(s) = m/s(s + m - 1) \). Finally, for odd \( s \), the map \( f:S^m \to S^{n(s)} \) is an embedding and, for even \( s \), it factors through the canonical projection \( \pi:S^m \to \mathbb{R}P^m \), yielding an embedding \( f:\mathbb{R}P^m \to S^{n(s)} \), \([4]\).

**Theorem 2** If \( s = 2 \) then for the standard minimal immersion \( f:S^m \to S^{n(2)} \) we have

\[
\dim PK(f) = \frac{1}{24} (5m^4 + 26m^3 + 19m^2 - 50m) - 1.
\]

In particular, \( f \) is infinitesimally rigid if and only if \( m = 2 \) (i.e. \( f:S^2 \to S^4 \) is the Veronese surface).

**Proof** The second statement follows from the first by direct computation using Proposition 1. To prove the first, we note that an element of \( H_{k(2)} \) is the restriction to \( S^m \) of a polynomial \( p:\mathbb{R}^{m+1} \to \mathbb{R} \) which has the form

\[
p = \sum_{k=1}^{m+1} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},
\]

where \( a_k, b_{ij} \in \mathbb{R} \) with

\[
\sum_{k=1}^{m+1} a_k = 0
\]

and

\( \varphi_k, \varphi_{ij} \) are scalars on \( S^m \) defined by \( \varphi_k(x) = x_k^2, \varphi_{ij}(x) = x_i x_j, x = (x_1, \ldots, x_{m+1}) \in S^m, k = 1, \ldots, m + 1, 1 \leq i < j \leq m + 1 \) (cf. \([2]\), p.176). The standard minimal immersion \( f:S^m \to S^{n(2)} \) is then defined by the more explicit form

\[
f(x_1, \ldots, x_{m+1}) = \frac{N}{1 - J} \sum_{k=1}^{m+1} \left( x_k^2 - \frac{1}{m + 1} \right) \varphi_k + \frac{2N}{J} \sum_{i<j} x_i x_j \varphi_{ij}, x \in S^m,
\]
where \( I = \| \varphi_k \|^2 \), \( J = \| \varphi_{ij} \|^2 \) and \( N > 0 \) is a normalizing factor given by the condition \( \| f \| = 1 \). A vector field \( \nu \) along \( f \) determines a map \( \tilde{\varphi} : S^m \to \mathcal{H}_{\lambda(2)} \) given by translating the vectors of \( \nu \) to the origin of \( \mathcal{H}_{\lambda(2)} \). Thus \( \tilde{\varphi} \) has the form

\[
\tilde{\varphi} = \sum_{k=1}^{m+1} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},
\]

where \( a_k, b_{ij} \) are scalars on \( S^m \) with

\[
\sum_{k=1}^{m+1} a_k = 0.
\]

Moreover, by [9], \( \nu \in K(f) \) is equivalent to \( \Delta \tilde{\varphi} = 2(m + 1) \nu \) since

\[
2e(f) = \text{trace} \| f_{ij} \|^2 = \frac{m}{k(2)} = 2(m + 1).
\]

Thus, orthogonality relations for \( \varphi_k \) and \( \varphi_{ij} \) ([2], p. 176) imply that \( a_k, b_{ij}, k = 1, \ldots, m + 1, 1 \leq i < j \leq m + 1, \) belong to \( \mathcal{H}_{\lambda(2)} \) and so we have

\[
a_r = \sum_{k=1}^{m+1} a_k^r \varphi_k + 2 \sum_{i<j} b_{ij}^r \varphi_{ij}, \quad r = 1, \ldots, m + 1,
\]

and

\[
b_{pq} = \sum_{k=1}^{m+1} a_k^{pq} \varphi_k + 2 \sum_{i<j} b_{ij}^{pq} \varphi_{ij}, \quad 1 \leq p < q \leq m + 1,
\]

where \( a_k^r, b_{ij}^r, a_k^{pq}, b_{ij}^{pq} \in \mathbb{R} \) such that

\[
(1) \quad \sum_{k=1}^{m+1} a_k^r = 0 \quad \text{and} \quad \sum_{k=1}^{m+1} a_k^{pq} = 0, \quad r = 1, \ldots, m + 1, 1 \leq p < q \leq m + 1,
\]

hold. The relation \( \sum_{k=1}^{m+1} a_k = 0 \) translates into

\[
(2) \quad \sum_{r=1}^{m+1} a_r^k = 0 \quad \text{and} \quad \sum_{r=1}^{m+1} b_{ij}^r = 0.
\]

(In particular, \( PK(f) = K(f) \).) Finally, as in the proof of Theorem 2 in [9], an easy computation shows that \( \langle f, \varphi \rangle = 0 \) is satisfied if and only if the following relations hold:

\[
(3) \quad a_k^k = 0, \quad k = 1, \ldots, m + 1;
\]

\[
(4) \quad a_i^i + a_j^j + 8b_{ij}^j = 0, \quad 1 \leq i < j \leq m + 1;
\]

\[
(5) \quad b_{ij}^i + 2a_{ij}^j = 0, \quad 1 \leq i < j \leq m + 1;
\]

\[
(6) \quad b_{ij}^i + 2a_{ij}^j = 0, \quad 1 \leq i < j \leq m + 1;
\]

\[
(7) \quad b_{ij}^i + 2a_{ij}^j + 4b_{ri}^j + 4b_{ri}^j = 0, \quad 1 \leq r < i < j \leq m + 1;
\]
(8) \[ b^r_{ij} + 2a^r_{ij} + 4b^r_{ij} + 4b^r_{ij} = 0, \quad 1 \leq i < r < j \leq m + 1; \]
(9) \[ b^r_{ij} + 2a^r_{ij} + 4b^r_{ij} + 4b^r_{ij} = 0, \quad 1 \leq i < j \leq m + 1; \]
(10) \[ b^p_{ij} + b^p_{ij} + b^p_{ij} + b^p_{ij} + b^p_{ij} + b^p_{ij} = 0 \]
for all \( 1 \leq i < j < p < q \leq m + 1 \).

Our task is to compute \( \dim K(f) \), i.e. the dimension of the linear space consisting of all solutions of the linear system (1)–(10). In order to solve the system (1)–(10) we first choose the coefficients \( a_k^r, k, r = 1, \ldots, m + 1 \), satisfying the corresponding relations in (1), (2) and (3). These span a linear space of dimension \( (m + 1)^2 - ((m + 1) + m + 1 + m) = m^2 - m - 1 \). The second relation of (2) yields that the linear space spanned by the coefficients \( b^r_{ij}, r = 1, \ldots, m + 1, 1 \leq i < j \leq m + 1 \), has dimension \( (m_2 + 1)m \). By relations (5)–(6) the coefficients \( a^r_{ij}, a^r_{ij}, 1 \leq i < j \leq m + 1 \), depend on \( a_k^r \). Hence, by the second relation of (1), each row
\[ a^r_{ij}, a^r_{ij}, \ldots, a^r_{ij}, 1 \leq i < j \leq m + 1, \]
contains exactly three dependent elements, i.e. the dimension of the linear space of independent coefficients \( a^r_{ij}, r = 1, \ldots, m + 1, 1 \leq i < j \leq m + 1 \), is \( (m - 2)(m_2 + 1) \). Now we have to determine the linear space of coefficients \( b^p_{ij}, 1 \leq i < j \leq m + 1, 1 \leq p < q \leq m + 1 \), which are independent from \( \{a_k^r, b^r_{ij}, a^r_{ij} \mid k, r = 1, \ldots, m + 1, 1 \leq i < j \leq m + 1 \} \). By relation (4) the elements \( b^r_{ij}, 1 \leq i < j \leq m + 1 \), are dependent. The linear space of independent coefficients \( b^p_{ij} \) such that \( i, j, p, q \) are not mutually distinct, by relations (7), (8) and (9), is spanned by \( b^p_{ij}, 1 \leq r < i < j \leq m + 1, b^p_{ij}, 1 \leq i < r < j \leq m + 1, b^p_{ij}, 1 \leq i < j < r \leq m + 1, \) i.e. its dimension is \( 3(m_3 + 1) \). Finally, the only relations between the coefficients \( b^p_{ij} \) for which \( i, j, p, q \) are mutually different are in (10). Each equation in (10) contains exactly one coefficient \( b^p_{ij} \) with \( i < j < p < q \) and so the dimension of the corresponding linear space is \( 5(m_4 + 1) \). Thus we have
\[
\dim K(f) = m^2 - m - 1 + (m_2 + 1)m + (m - 2)(m_2 + 1) \\
+ 3(m_3 + 1) + 5(m_4 + 1).
\]
and a straightforward computation completes the proof. \( \square \)

Next we turn to the local rigidity of the standard minimal immersions. As has already been established in [9], the Veronese surface \( f:S^2 \to S^4 \) has trivial variation space, in particular, it is locally rigid. In contrast to this, for \( m \geq 3 \), we have the following:

**Theorem 3** If \( m \geq 3 \) is odd, then the standard minimal immersion \( f:S^m \to S^{m(s)} \) is non-locally rigid.
Flexible harmonic maps into spheres

Proof First we show that local rigidity of \( f: S^m \to S^{n(s)} \) implies that \( f \) totally geodesic, i.e. it maps geodesics of \( S^m \) onto geodesics of \( S^{n(s)} \) linearly [6]. Let \( \gamma: \mathbb{R} \to S^m \) be a geodesic with initial vector \( X_{x_0} \in T_{x_0}(S^m) \), \( x_0 = \gamma(0) \). Because \( m \) is odd we can extend the vector \( X_{x_0} \) to an infinitesimal isometry \( X \in \text{so}(m + 1) \) with \( ||X|| = \text{const} \). Then, denoting by \( (\varphi_t) \subset SO(m + 1) \) the one-parameter group of isometries induced by \( X \), the integral curves \( \varphi_t(x), x \in S^m \), are geodesics, in particular the vector field \( \nu = f_*(X) \) along \( f \) is projectable. On the other hand, by the lemma, \( \nu \in K(f) \) and, since

\[
||\nu|| = ||f_*(X)|| = \frac{1}{\sqrt{k(s)}} ||x|| = \text{const},
\]

we get \( \nu \in V(f) \). Local rigidity of \( f \) implies the existence of a one-parameter group \( (\varphi_t) \subset SO(n(s) + 1) \) of isometries such that \( f_t = \exp \circ (t\nu) = \psi_t \) holds for all \( t \in \mathbb{R} \). Thus, if \( Y \) denotes the infinitesimal isometry induced by \( (\psi_t) \) then \( f_*(X) = Y \circ f \), i.e. the vector fields \( X \) and \( Y \) are \( f \)-related. Follows that \( f \circ \varphi_t = \psi_t \circ f \) is valid for all \( t \in \mathbb{R} \), i.e. the geodesic \( t \mapsto \gamma(t) \varphi_t(x_0) \) is mapped under \( f \) to the geodesic \( t \mapsto \exp(tv_{x_0}) \) since \( f(\varphi_t(x_0)) = \exp(tv_{x_0}), t \in \mathbb{R} \), and hence \( f \) is totally geodesic. Clearly, the image of \( f \) is then a totally geodesic submanifold of \( S^{n(s)} \) which contradicts the fullness of \( f \). Thus the theorem is proved. \( \square \)

The only property of the domain used in the proof of Theorem 3 is the extendability of a given vector \( X_{x_0} \) to an infinitesimal isometry \( X \) with \( ||X|| = \text{const} \). Thus, by the same argument, we get the following:

Corollary 3 Let \( M \) be either an odd Euclidean sphere or a compact Lie group with bi-invariant metric. Then any full harmonic homothetic embedding \( f:M \to S^n \), with \( \text{dim} M < n \), is non-locally rigid.

In particular, the harmonic embedding \( f:T^2 \to S^3 \) of Example 2 is non-locally rigid.

Remark According to Calabi [3] and Do Carmo and Wallach [4] full isometric minimal immersions of the sphere \( S^m \) of constant sectional curvature \( k(s) \) into \( S^{n(s)} \) are rigid, provided that \( s \leq 3 \), i.e. for any two such maps \( f,f' : S^m \to S^{n(s)} \) there exists an isometry \( \varphi \in O(n(s) + 1) \) with \( \varphi \circ f = f' \). On the other hand, non-local rigidity of the standard minima immersion \( f:S^m \to S^{n(s)} \), for odd \( m \) and \( s \leq 3 \), and the rigidity theorem above imply the existence of a harmonic variation \( \nu \in V(f) \) along which the deformed harmonic maps \( f_t = \exp \circ (t\nu) \) will not be in general homothetic. Thus we have examples of homothetic minimal immersions which are rigid among homothetic minimal immersions but non-rigid among harmonic maps.
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REFERENCES