G. TÓTH AND G. D'AMBRA

PARAMETER SPACE FOR HARMONIC MAPS OF CONSTANT ENERGY DENSITY INTO SPHERES

1. Introduction and Statement of the Results

As a generalization of the Do Carmo–Wallach description of minimal immersions into spheres (see [4] and [9]) the object of this paper is to classify harmonic maps of constant energy density of a fixed Riemannian manifold M (of dimension >1) into Euclidean n-spheres, $n \in \mathbb{N}$. A map $f: M \to S^n$ is harmonic [6] if the vector function $\bar{f} = (f^0, \ldots, f^n): M \to \mathbb{R}^{n+1}$ induced by f via the inclusion $S^n \subset \mathbb{R}^{n+1}$ statisfies the equation

$$\Delta \bar{f} = 2e(f) \cdot \bar{f},$$

where Δ is the Laplacian on M and the scalar $e(f)$ on M stands for the energy density of f. In particular, if $e(f) = \lambda/2$ is constant, then $\lambda \in \mathrm{Spec}(M)$ ($=$ spectrum of M) and every component f^i, $i = 0, \ldots, n$, belongs to the eigenspace V_λ of Δ corresponding to λ.

Given a harmonic map $f: M \to S^n$, denote by $K(f)$ the vector space of divergence-free Jacobi fields along f [8]. Then [8] $\mathrm{so}(n+1) \circ f \subset PK(f)$, where $\mathrm{so}(n+1)$ is the Lie algebra of Killing vector fields on S^n and $PK(f) \subset K(f)$ stands for the linear subspace of projectable elements, i.e. $PK(f) = \{v \in K(f) \mid v_x = v_{x'} \text{ whenever } f(x) = f(x'), x, x' \in M\}$. Furthermore [7], $v \in K(f)$ if and only if

$$\Delta \hat{v} = 2e(f) \cdot \hat{v},$$

where $\hat{v} = (v^0, \ldots, v^n): M \to \mathbb{R}^{n+1}$ is the vector function obtained from v by translating tangent vectors of $S^n \subset \mathbb{R}^{n+1}$ to the origin via the canonical identification $\hat{\ }: T(\mathbb{R}^{n+1}) \to \mathbb{R}^{n+1}$. Especially, if $e(f) = \lambda/2$, $\lambda \in \mathrm{Spec}(M)$, then $v^i \in V_\lambda$, $i = 0, \ldots, n$. For fixed $n \in \mathbb{N}$, the orthogonal group $O(n+1)$ acts, by composition, on the space of all harmonic maps $f: M \to S^n$ with energy density $e(f) = \lambda/2 \in \mathbb{R}$ and, passing to the orbit space, we are led to study the equivalence classes of harmonic maps of M into S^n, where two maps $f, f': M \to S^n$ are said to be equivalent if there exists $U \in O(n+1)$ such that $f' = U \circ f$. For the following classification theorem to be proved in Section 2, recall that a map $f: M \to S^n$ is said to be full if $\mathrm{im}(\bar{f}) \subset \mathbb{R}^{n+1}$ is not contained in a proper linear subspace of $\mathbb{R}^{n+1}$.

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THEOREM 1. *Let G be a compact Lie group, $K \subset G$ a closed subgroup and assume that $M = G/K$ is a (compact) oriented (isotropy) irreducible homogeneous space with invariant Riemannian metric g. Given $(0 \neq)\lambda \in \mathrm{Spec}(G/K)$, the equivalence classes of full harmonic maps $f\colon G/K \to S^n$ with $e(f) = \lambda/2$ can be (smoothly) parametrized by a compact convex body L lying in a finite dimensional vector space E. The interior points of L correspond to maps with maximal $n(= n(\lambda) = \dim V_\lambda - 1)$. Finally, for every full harmonic map $f\colon G/K \to S^{n(\lambda)}$ with $e(f) = \lambda/2$, we have $PK(f) = K(f)$ and $K(f)/\mathrm{so}(n(\lambda) + 1) \circ f \cong E$.*

REMARK 1. Recall [8] that a harmonic map $f\colon M \to S^n$ is said to be infinitesimally rigid if $\mathrm{so}(n+1) \circ f = PK(f)$. Keeping the hypotheses of Theorem 1 we obtain that a full harmonic map $f\colon G/K \to S^{n(\lambda)}$ with $e(f) = \lambda/2$ is infinitesimally rigid if and only if $L = E = \{0\}$, or equivalently, if for every harmonic map $f'\colon G/K \to S^{n(\lambda)}$ with $e(f') = \lambda/2$ there exists $U \in O(n(\lambda) + 1)$ such that $f' = U \circ f$.

REMARK 2. From the proof of Theorem 1 it follows that the space L^0 parametrizing the (equivalence classes of) full minimal isometric immersions $f\colon G/K \to S^n$ with induced Riemannian metric $(\lambda/m)g$ is the intersection of L with a linear subspace of E. Combining this with Remark 1.1 we obtain that if a full minimal isometric immersion $f\colon G/K \to S^{n(\lambda)}$ with induced metric $(\lambda/m)g$ is infinitesimally rigid (as a harmonic map with energy density $\lambda/2$) then f is (linearly) rigid in the sense of [9].

Using the Do Carmo–Wallach theory, in Section 3 we determine dim $L = \dim E$ for a spherical domain $G/K = \mathrm{SO}(m+1)/\mathrm{SO}(m) = S^m$.

THEOREM 2. *Let $G/K = \mathrm{SO}(m+1)/\mathrm{SO}(m) = S^m$ and $\lambda_k = k(k+m-1) \in \mathrm{Spec}(S^m)$, $k \in \mathbb{N}$. If $m = 2$ or $k = 1$, we have $\dim E = 0$. Furthermore, denoting by V_m^σ the irreducible complex $\mathrm{SO}(m+1)$-module (= representation space for $\mathrm{SO}(m+1)$) with highest weight $\sigma = (\sigma_1, \ldots, \sigma_l) \in (\frac{1}{2}\mathbb{Z})^l$, $l = [(m+1)/2]$, if $k > 1$, we have, for $m = 3$,*

$$(*) \qquad \dim E = \sum_{\substack{(a,b)\in\Delta \\ a,b\ \mathrm{even}}} \dim_{\mathbb{C}} \{V_3^{(a,b)} \oplus V_3^{(a,-b)}\}$$

and, for $m > 3$,

$$(**) \qquad \dim E = \sum_{\substack{(a,b)\in\Delta \\ a,b\ \mathrm{even}}} \dim_{\mathbb{C}} V_m^{(a,b,0,\ldots,0)},$$

where $\Delta \subset \mathbb{R}^2$ is the closed (convex) triangle with vertices $(2, 2)$, (k, k) and $(2k-2, 2)$.

REMARK 3. As shown in Section 3, $\dim_{\mathbb{C}} V_m^{(a,b,0,\ldots,0)}$, occurring in (*)–(**), can be determined by the Weyl dimension formula [2, p. 266]).

REMARK 4. In [4] Do Carmo and Wallach gave a lower estimate for $\dim L^0$ which is, for harmonic maps, replaced here by the exact determination of dim $L(\geqslant \dim L^0)$. In particular, by [9] and Theorem 2, for $m > 2$ and $k = 2, 3$, we have $\dim L > \dim L^0 = 0$ and, for $m > 2$ and $k > 3$, $\dim L \geqslant \dim L^0 \geqslant 18$.

REMARK 5. For $m = 2$, Theorem 2 and Remark 1 yield Calabi's rigidity theorem [3]. For generalities on harmonic maps, the Report [5] serves as a general reference and, for the Do Carmo–Wallach theory of minimal immersions, we use the results of [4] and [9].

2. PROOF OF THE CLASSIFICATION THEOREM

Let G/K be a compact oriented irreducible homogeneous space with invariant Riemannian metric g and origin $O = \{K\} \in G/K$. For fixed $(O \neq)\lambda \in \mathrm{Spec}(G/K)$, define a scalar product $\langle\ ,\ \rangle$ on the eigenspace $V_\lambda \subset C^\infty(G/K)$ corresponding to λ by

$$\langle \mu, \mu' \rangle = \frac{n(\lambda) + 1}{\int_{G/K} \mathrm{vol}(G/K, g)} \int_{G/K} \mu \cdot \mu' \,\mathrm{vol}(G/K, g),$$

where $\dim V_\lambda = n(\lambda) + 1$ and $\mathrm{vol}(G/K, g)$ stands for the volume form on G/K.

The canonical action of G on G/K (by isometries with respect to g) gives rise to a (linear) representation of G on $C^\infty(G/K)$ by setting $a \cdot \mu = \mu \circ a^{-1}$, $a \in G$, $\mu \in C^\infty(G/K)$. This leaves the eigenspace $V_\lambda \in C^\infty(G/K)$ invariant and preserves the scalar product $\langle\ ,\ \rangle$ on V_λ, i.e. we obtain an orthogonal representation $\rho: G \to \mathrm{SO}(V_\lambda)$.

For fixed orthonormal base $\{f_\lambda^i\}_{i=0}^{n(\lambda)} \subset V_\lambda$ which, at the same time, identifies V_λ with $\mathbb{R}^{n(\lambda)+1}$, define a map

$$\tilde{f}_\lambda: G/K \to V_\lambda (= \mathbb{R}^{n(\lambda)+1})$$

by

$$\tilde{f}_\lambda(x) = \sum_{i=0}^{n(\lambda)} f_\lambda^i(x) f_\lambda^i = (f_\lambda^0(x), \ldots, f_\lambda^{n(\lambda)}(x)), \qquad x \in G/K.$$

Then [4] $\mathrm{im}(\tilde{f}_\lambda) \subset S^{n(\lambda)}$ and the induced map $f_\lambda: G/K \to S^{n(\lambda)}$ is a minimal immersion with induced Riemannian metric $(\lambda/m)g$ or, keeping the original metric g on G/K, f_λ is a full harmonic (homothetic) immersion with energy density $e(f_\lambda) = \lambda/2$. The map f_λ is said to be the standard minimal immersion associated to the eigenvalue $\lambda \in \mathrm{Spec}(G/K)$. (Clearly, different choices of the orthonormal base in V_λ give rise to equivalent standard minimal immersions.) The identification $V_\lambda = \mathbb{R}^{n(\lambda)+1}$ above translates the orthogonal representation ρ to a matrix representation $\rho: G \to \mathrm{SO}(n(\lambda)+1)$ such that $f_\lambda: G/K \to S^{n(\lambda)}$ is equivariant with respect to ρ, i.e. we have $f_\lambda \circ a = \rho(a) \circ f_\lambda$, $a \in G$. Clearly, $v^0 = \bar{f}_\lambda(0) \in V_\lambda$ is left fixed by $\rho(K)$.

Let W^0 denote the linear subspace of the symmetric square $S^2(V_\lambda) = S^2(\mathbb{R}^{n(\lambda)+1})$ given by

$$
\begin{aligned}
W^0 &= \mathrm{span}_{\mathbb{R}}\{\rho(a)((v^0)^2) \in S^2(V_\lambda) \mid a \in G\} \\
&= \mathrm{span}_{\mathbb{R}}\{(\bar{f}_\lambda(x))^2 \in S^2(\mathbb{R}^{n(\lambda)+1}) \mid x \in G/K\}
\end{aligned}
$$

and set

$$E = (W^0)^\perp \subset S^2(\mathbb{R}^{n(\lambda)+1}),$$

where the orthogonal complement is taken with respect to the scalar product $\langle A, B \rangle = \mathrm{trace}\, B^t \circ A$, $A, B \in S^2(\mathbb{R}^{n(\lambda)+1})$ (t = transpose). Finally, let $L \subset E$ be the convex body defined by

$$L = \{C \in E \mid C + I_{n(\lambda)+1} \text{ is positive semidefinite}\},$$

where $I_{n(\lambda)+1}$ = identity of $\mathbb{R}^{n(\lambda)+1}$.

Given a full harmonic map $f: G/K \to S^n$ with $e(f) = \lambda/2$, the system $\{f^i\}_{i=0}^n$ (of components of $\bar{f}$) is a linearly independent set in V_λ; in particular, $n \leqslant n(\lambda)$. By polar decomposition of matrices, there exists a positive semidefinite endomorphism $B \in S^2(\mathbb{R}^{n(\lambda)+1})$ such that $\mathrm{im}(B \cdot \bar{f}_\lambda) \subset S^{n(\lambda)}$ and $i \circ f$ is equivalent to $B \circ f_\lambda$, where $i: S^n \to S^{n(\lambda)}$ denotes the canonical inclusion map. Moreover, B is uniquely determined by the equivalence class of f. Associate then to f the matrix $C = B^2 - I_{n(\lambda)+1}$. As $B \circ f_\lambda$ maps into $S^{n(\lambda)}$, for $x \in G/K$, we have

$$\langle C, (\bar{f}_\lambda(x))^2 \rangle = \langle C \cdot \bar{f}_\lambda(x), \bar{f}_\lambda(x) \rangle = \langle B \cdot \bar{f}_\lambda(x), B \cdot \bar{f}_\lambda(x) \rangle - 1 = 0$$

and hence $C \in L$. Now, by the same argument as in the proof of the Classification Theorem in [9], we obtain that the correspondence $f \to C$ gives rise to a parametrization of the space of equivalence classes of full harmonic maps $f: G/K \to S^n$ with $e(f) = \lambda/2$ by the convex body $L \subset E$. Moreover, L is compact.

To prove the last statement of Theorem 1, let $f: G/K \to S^{n(\lambda)}$ be a full harmonic map with $e(f) = \lambda/2$. By Section 1, $v \in K(f)$ if and only if $\{v^i\}_{i=0}^{n(\lambda)} \subset V_\lambda$ with $\langle \bar{f}, \hat{v} \rangle = 0$. As f is full there exists a unique $(n(\lambda)+1) \times (n(\lambda)+1)$ matrix X such that $\hat{v} = X \cdot \bar{f}$, especially, $K(f) = PK(f)$. Writing $X = A + B$,

$A \in \mathrm{so}(n(\lambda)+1)$ and $B \in S^2(\mathbb{R}^{n(\lambda)+1})$, the relation

$$\langle \bar{f}, \hat{v} \rangle = \langle \bar{f}, A \cdot \bar{f} \rangle + \langle \bar{f}, B \cdot \bar{f} \rangle = 0$$

splits into $\langle \bar{f}, A \cdot \bar{f} \rangle = \langle \bar{f}, B \cdot \bar{f} \rangle = 0$. By fullness of f, there exists a non-singular $(n(\lambda)+1) \times (n(\lambda)+1)$ matrix Y with $\bar{f} = Y \cdot \bar{f}_\lambda$. Then, $Y^t \cdot B \cdot Y \in S^2(\mathbb{R}^{n(\lambda)+1})$ and, for $x \in G/K$, we have

$$\begin{aligned} O = \langle \bar{f}(x), B \cdot \bar{f}(x) \rangle &= \langle Y \cdot \bar{f}_\lambda(x), B \cdot Y \cdot \bar{f}_\lambda(x) \rangle \\ &= \langle \bar{f}_\lambda(x), (Y^t \cdot B \cdot Y) \bar{f}_\lambda(x) \rangle \\ &= \langle (Y^t \cdot B \cdot Y), (\bar{f}_\lambda(x))^2 \rangle, \end{aligned}$$

i.e. $Y^t \cdot B \cdot Y \in E$. Now, the correspondence which associates to $v \in K(f)$ the pair $(A \circ f, Y^t \cdot B \cdot Y) \in (\mathrm{so}(n(\lambda)+1) \circ f) \oplus E$ is a linear isomorphism. Hence $K(f)/\mathrm{so}(n(\lambda)+1) \circ f \cong E$, which completes the proof of Theorem 1.

REMARK 6. Setting

$$W = \mathrm{span}_{\mathbb{R}} \{ S^2((\bar{f}_\lambda)_*(T_x(G/K)))^\wedge \mid x \in G/K \} \subset S^2(\mathbb{R}^{n(\lambda)+1})$$

and using the notations of Section 1, by [4], we have $W^0 \subset W$ and $L^0 = L \cap W^\perp$.

3. Computation of $\dim L$ for spherical domains

Let $G = \mathrm{SO}(m+1)$, $K = \mathrm{SO}(m)$ and endow $G/K = S^m$ with the Euclidean metric. Then [1] $\mathrm{Spec}(S^m) = \{\lambda_k = k(k+m-1) \mid k \in \mathbb{Z}_+\}$ and, for each $k \in \mathbb{Z}_+$, the eigenspace V_{λ_k} corresponding to λ_k is the vector space $\mathscr{H}_m^k$ of spherical harmonics of order k on S^m with

$$\dim \mathscr{H}_m^k = n(\lambda_k) + 1 = (2k+m-1)\frac{(k+m-2)!}{k!(m-1)!}.$$

Furthermore, as an orthogonal $\mathrm{SO}(m+1)$-module, $\mathscr{H}_m^k$ is irreducible [1] and, for fixed orthonormal base $\{f_{\lambda_k}^i\}_{i=0}^{n(\lambda_k)} \subset \mathscr{H}_m^k$, the construction of the standard minimal immersion $f_{\lambda_k}: S^m \to S^{n(\lambda_k)}$, in Section 2, shows that the $\mathrm{SO}(m+1)$-module structure ρ on $\mathscr{H}_m^k$ is also class 1 for the pair $(\mathrm{SO}(m+1), \mathrm{SO}(m))$ – i.e. there exists a unit vector $v^0 \in \mathscr{H}_m^k$ left fixed by $\rho(\mathrm{SO}(m))$. (Here and in what follows we use the notions and results of [9] without making explicit references.) Conversely, every class 1 representation of $(\mathrm{SO}(m+1), \mathrm{SO}(m))$ is equivalent to some $\mathscr{H}_m^k$ (considered as an irreducible orthogonal $\mathrm{SO}(m+1)$-module.) Denoting also by ρ the induced representation on the symmetric square $S^2(\mathscr{H}_m^k)$, the $\mathrm{SO}(m+1)$-submodule

$$W^0 = \mathrm{span}_{\mathbb{R}} \{ \rho(a)((v^0)^2) \in S^2(\mathscr{H}_m^k) \mid a \in \mathrm{SO}(m+1) \}$$

of $S^2(\mathscr{H}_m^k)$ is the sum of all submodules which are class 1 for $(\mathrm{SO}(m+1), \mathrm{SO}(m))$.

For $m=2$, by elementary representation theory, each submodule of $S^2(\mathscr{H}_2^k)$ is class 1 for $(\mathrm{SO}(3), \mathrm{SO}(2))$ (since $\dim \mathscr{H}_2^k = 2k+1$ is odd). Thus, $W^0 = S^2(\mathscr{H}_2^k)$ and hence $(W^0)^\perp = E = \{0\}$. For $k=1$, a standard minimal immersion $f_{\lambda_1}: S^m \to S^m$ is nothing but an isometry and hence infinitesimally rigid [8], i.e. by Theorem 1, $E = \{0\}$.

Setting $m>2$ and $k>1$, we now prove (*)–(**). As irreducibility of submodules in $S^2(\mathscr{H}_m^k)$ do not depend on field extensions, the complexification $W^0 \otimes_{\mathbb{R}} \mathbb{C}$ is the sum of all irreducible complex $\mathrm{SO}(m+1)$-modules in $S^2(\mathscr{H}_m^k \otimes_{\mathbb{R}} \mathbb{C})$ which are class 1 for $(\mathrm{SO}(m+1), \mathrm{SO}(m))$. By a result of Do Carmo–Wallach in [4] we have the following decompositions

$$(*') \qquad S^2(\mathscr{H}_3^k \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{j=0}^{[k/2]} \{V_3^{(2k-2j,2j)} \oplus V_3^{(2k-2j,-2j)}\} \oplus S^2(\mathscr{H}_3^{k-1} \otimes_{\mathbb{R}} \mathbb{C})$$

and

$$(**') \qquad S^2(\mathscr{H}_m^k \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{j=0}^{[k/2]} V_m^{(2k-2j,2j,0,\ldots,0)} \oplus S^2(\mathscr{H}_m^{k-1} \otimes_{\mathbb{R}} \mathbb{C}), \quad m>3,$$

where V_m^σ stands for the irreducible complex $\mathrm{SO}(m+1)$-module with highest weight $\sigma = (\sigma_1, \ldots, \sigma_l) \in (\frac{1}{2}\mathbb{Z})^l$, $l = [(m+1)/2]$. Since $V_m^{(i,0,\ldots,0)} = \mathscr{H}_m^i \otimes_{\mathbb{R}} \mathbb{C}$, $i \in \mathbb{Z}_+$, (*′) and (**′) imply (*) and (**), resp.

REMARK 7. Once $\dim_{\mathbb{C}} V_m^{(a,b,0,\ldots,0)}$ is known for each $(a,b) \in \Delta$, a, b even, we can compute $\dim L = \dim E$ via (*)–(**). In what follows, using the Weyl dimension formula [2] we determine $\dim_{\mathbb{C}} V_m^{(a,b,0,\ldots,0)}$.

(i) $m = 2l$ even, $l>2$. We have

$$\begin{aligned}
\dim_{\mathbb{C}} & V_m^{(a,b,0,\ldots,0)} \\
&= \frac{(a-b+1)(a+b+2l-2)}{2l-2} \times \prod_{r=3}^{l} \frac{(a+r-1)(a+2l-r)}{(r-1)(2l-r)} \\
&\quad \times \prod_{r=3}^{l} \frac{(b+r-2)(b+2l-r-1)}{(r-2)(2l-r-1)} \times \frac{(2a+2l-1)(2b+2l-3)}{(2l-1)(2l-3)} \\
&= \frac{(a-b+1)(a+b+2l-2)}{2l-2} \times \frac{1}{a+1} \binom{a+2l-3}{a} \binom{b+2l-4}{b} \\
&\quad \times \frac{(2a+2l-1)(2b+2l-3)}{(2l-1)(2l-3)} \\
&= \frac{(a-b+1)(a+b+m-2)(2a+m-1)(2b+m-3)}{(a+1)(m-1)(m-2)(m-3)}
\end{aligned}$$

$$\times\binom{a+m-3}{a}\binom{b+m-4}{b}.$$

(ii) $m=2l-1$ odd, $l>2$. By similar computations we obtain the same formula for $\dim_{\mathbb{C}} V_m^{(a,b,0,\ldots,0)}$ as above.

(iii) $m=3,4$. Again by the Weyl dimension formula

$$\dim_{\mathbb{C}} V_3^{(a,b)} = \dim_{\mathbb{C}} V_3^{(a,-b)} = (a-b+1)(a+b+1)$$

and

$$\dim_{\mathbb{C}} V_4^{(a,b)} = \tfrac{1}{6}(a-b+1)(a+b+2)(2a+3)(2b+1).$$

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Authors' addresses:

G. Tóth,
The Ohio State University,
Department of Mathematics,
231 West 18th Avenue,
Columbus, OH 43210
U.S.A.

G. D'Ambra,
Istituto Matematico,
Università di Cagliari,
Via Ospedale 72,
09100 Cagliari,
Italy

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