1. Introduction and statement of the results

As a generalization of the Do Carmo–Wallach description of minimal immersions into spheres (see [4] and [9]) the object of this paper is to classify harmonic maps of constant energy density of a fixed Riemannian manifold $M$ (of dimension $> 1$) into Euclidean $n$-spheres, $n \in \mathbb{N}$. A map $f : M \to S^n$ is harmonic [6] if the vector function $\overline{f} = (f^0, \ldots, f^n) : M \to \mathbb{R}^{n+1}$ induced by $f$ via the inclusion $S^n \subset \mathbb{R}^{n+1}$ satisfies the equation
\[
\Delta \overline{f} = 2e(f) \cdot \overline{f},
\]
where $\Delta$ is the Laplacian on $M$ and the scalar $e(f)$ on $M$ stands for the energy density of $f$. In particular, if $e(f) = \lambda/2$ is constant, then $\lambda \in \text{Spec}(M)$ (= spectrum of $M$) and every component $f^i$, $i = 0, \ldots, n$, belongs to the eigenspace $V_\lambda$ of $\Delta$ corresponding to $\lambda$.

Given a harmonic map $f : M \to S^n$, denote by $K(f)$ the vector space of divergence-free Jacobi fields along $f$ [8]. Then [8] $\text{so}(n+1) \cdot f \subset PK(f)$, where $\text{so}(n+1)$ is the Lie algebra of Killing vector fields on $S^n$ and $PK(f) \subset K(f)$ stands for the linear subspace of projectable elements, i.e. $PK(f) = \{ v \in K(f) | v_x = v_x \text{ whenever } f(x) = f(x') \}, x, x' \in M \}$. Furthermore [7], $v \in PK(f)$ if and only if
\[
\Delta v = 2e(f) \cdot v,
\]
where $\dot{v} = (v^0, \ldots, v^n) : M \to \mathbb{R}^{n+1}$ is the vector function obtained from $v$ by translating tangent vectors of $S^n \subset \mathbb{R}^{n+1}$ to the origin via the canonical identification $\tilde{\cdot} : T(\mathbb{R}^{n+1}) \to \mathbb{R}^{n+1}$. Especially, if $e(f) = \lambda/2$, $\lambda \in \text{Spec}(M)$, then $v^i \in V_\lambda$, $i = 0, \ldots, n$. For fixed $n \in \mathbb{N}$, the orthogonal group $O(n+1)$ acts, by composition, on the space of all harmonic maps $f : M \to S^n$ with energy density $e(f) = \lambda/2 \in \mathbb{R}$ and, passing to the orbit space, we are led to study the equivalence classes of harmonic maps of $M$ into $S^n$, where two maps $f, f' : M \to S^n$ are said to be equivalent if there exists $U \in O(n+1)$ such that $f' = U \circ f$. For the following classification theorem to be proved in Section 2, recall that a map $f : M \to S^n$ is said to be full if $\text{im}(f) \subset \mathbb{R}^{n+1}$ is not contained in a proper linear subspace of $\mathbb{R}^{n+1}$.

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THEOREM 1. Let $G$ be a compact Lie group, $K \subset G$ a closed subgroup and assume that $M = G/K$ is a (compact) oriented (isotropy) irreducible homogeneous space with invariant Riemannian metric $g$. Given $(0 \neq) \lambda \in \text{Spec}(G/K)$, the equivalence classes of full harmonic maps $f: G/K \to S^n$ with $e(f) = \lambda/2$ can be (smoothly) parametrized by a compact convex body $L$ lying in a finite dimensional vector space $E$. The interior points of $L$ correspond to maps with maximal $n = n(\lambda) = \dim V_{\lambda} - 1$. Finally, for every full harmonic map $f: G/K \to S^{n(\lambda)}$ with $e(f) = \lambda/2$, we have $PK(f) = K(f)$ and $K(f)/\text{so}(n(\lambda) + 1)$ $f \cong E$.

REMARK 1. Recall [8] that a harmonic map $f: M \to S^n$ is said to be infinitesimally rigid if $\text{so}(n + 1) f = PK(f)$. Keeping the hypotheses of Theorem 1 we obtain that a full harmonic map $f: G/K \to S^{n(\lambda)}$ with $e(f) = \lambda/2$ is infinitesimally rigid if and only if $L = E = \{0\}$, or equivalently, if for every harmonic map $f': G/K \to S^{n(\lambda)}$ with $e(f') = \lambda/2$ there exists $U \in O(n(\lambda) + 1)$ such that $f' = U \circ f$.

REMARK 2. From the proof of Theorem 1 it follows that the space $L^0$ parametrizing the (equivalence classes of) full minimal isometric immersions $f: G/K \to S^n$ with induced Riemannian metric $(\lambda/m) g$ is the intersection of $L$ with a linear subspace of $E$. Combining this with Remark 1.1 we obtain that if a full minimal isometric immersion $f: G/K \to S^{n(\lambda)}$ with induced metric $(\lambda/m) g$ is infinitesimally rigid (as a harmonic map with energy density $\lambda/2$) then $f$ is (linearly) rigid in the sense of [9].

Using the Do Carmo–Wallach theory, in Section 3 we determine $\dim L = \dim E$ for a spherical domain $G/K = \text{SO}(m + 1)/\text{SO}(m) = S^m$.

THEOREM 2. Let $G/K = \text{SO}(m + 1)/\text{SO}(m) = S^m$ and $\lambda_k = k(k + m - 1) \in \text{Spec}(S^m)$, $k \in \mathbb{N}$. If $m = 2$ or $k = 1$, we have $\dim E = 0$. Furthermore, denoting by $V^a$ the irreducible complex $\text{SO}(m + 1)$-module (= representation space for $\text{SO}(m + 1)$) with highest weight $\sigma = (\sigma_1, \ldots, \sigma_l) \in (\frac{1}{2} \mathbb{Z})^l$, $l = [(m + 1)/2]$, if $k > 1$, we have, for $m = 3$,

\[(*)\quad \dim E = \sum_{(a, b) \in \Delta} \dim_c \{V_3^{(a, b)} \oplus V_3^{(a, -b)}\}\]

and, for $m > 3$,

\[(**)\quad \dim E = \sum_{(a, b) \in \Delta} \dim_c V_m^{(a, b, 0, \ldots, 0)},\]

where $\Delta \subset \mathbb{R}^2$ is the closed (convex) triangle with vertices $(2, 2), (k, k)$ and $(2k - 2, 2)$. 

REMARK 3. As shown in Section 3, \( \dim_c V_{m}^{(a,b,0,\ldots,0)} \), occurring in \((\ast)-(\ast\ast)\), can be determined by the Weyl dimension formula [2, p. 266]).

REMARK 4. In [4] Do Carmo and Wallach gave a lower estimate for \( \dim L^0 \) which is, for harmonic maps, replaced here by the exact determination of \( \dim L \geq \dim L^0 \). In particular, by [9] and Theorem 2, for \( m > 2 \) and \( k = 2, 3 \), we have \( \dim L > \dim L^0 = 0 \) and, for \( m > 2 \) and \( k > 3 \), \( \dim L > \dim L^0 \geq 18 \).

REMARK 5. For \( m = 2 \), Theorem 2 and Remark 1 yield Calabi's rigidity theorem [3]. For generalities on harmonic maps, the Report [5] serves as a general reference and, for the Do Carmo–Wallach theory of minimal immersions, we use the results of [4] and [9].

2. PROOF OF THE CLASSIFICATION THEOREM

Let \( G/K \) be a compact oriented irreducible homogeneous space with invariant Riemannian metric \( g \) and origin \( O = \{K\} \in G/K \). For fixed \( \lambda \neq \lambda \in \text{Spec}(G/K) \), define a scalar product \( \langle , \rangle \) on the eigenspace \( V_\lambda \subset C^\infty(G/K) \) corresponding to \( \lambda \) by

\[
\langle \mu, \mu' \rangle = \frac{n(\lambda) + 1}{\int_{G/K} \text{vol}(G/K,g)} \int_{G/K} \mu \cdot \mu' \text{vol}(G/K,g),
\]

where \( \dim V_\lambda = n(\lambda) + 1 \) and \( \text{vol}(G/K,g) \) stands for the volume form on \( G/K \).

The canonical action of \( G \) on \( G/K \) (by isometries with respect to \( g \)) gives rise to a (linear) representation of \( G \) on \( C^\infty(G/K) \) by setting \( a \cdot \mu = \mu a^{-1}, \ a \in G, \ \mu \in C^\infty(G/K) \). This leaves the eigenspace \( V_\lambda \subset C^\infty(G/K) \) invariant and preserves the scalar product \( \langle , \rangle \) on \( V_\lambda \), i.e. we obtain an orthogonal representation \( \rho: G \to \text{SO}(V_\lambda) \).

For fixed orthonormal base \( \{f_\lambda^i\}_{i=0}^{n(\lambda)} \subset V_\lambda \) which, at the same time, identifies \( V_\lambda \) with \( \mathbb{R}^{n(\lambda)+1} \), define a map

\[
\tilde{f}_\lambda: G/K \to V_\lambda (= \mathbb{R}^{n(\lambda)+1})
\]

by

\[
\tilde{f}_\lambda(x) = \sum_{i=0}^{n(\lambda)} f_\lambda^i(x) f_\lambda^i = (f_\lambda^0(x), \ldots, f_\lambda^{n(\lambda)}(x)), \quad x \in G/K.
\]

Then [4] \( \text{im}(\tilde{f}_\lambda) \subset S^{n(\lambda)} \) and the induced map \( f_\lambda: G/K \to S^{n(\lambda)} \) is a minimal immersion with induced Riemannian metric \( (\lambda/m)g \) or, keeping the original metric \( g \) on \( G/K \), \( f_\lambda \) is a full harmonic (homothetic) immersion with energy density \( e(f_\lambda) = \lambda/2 \). The map \( f_\lambda \) is said to be the standard minimal immersion.
associated to the eigenvalue $\lambda \in \text{Spec}(G/K)$. (Clearly, different choices of the orthonormal base in $V_\lambda$ give rise to equivalent standard minimal immersions.) The identification $V_\lambda = \mathbb{R}^{n(\lambda)+1}$ above translates the orthogonal representation $\rho$ to a matrix representation $\rho: G \to \text{SO}(n(\lambda) + 1)$ such that $f_\lambda: G/K \to S^{n(\lambda)}$ is equivariant with respect to $\rho$, i.e. we have $f_\lambda \circ a = \rho(a) \circ f_\lambda$, $a \in G$. Clearly, $v^0 = f_\lambda(0) \in V_\lambda$ is left fixed by $\rho(K)$.

Let $W^0$ denote the linear subspace of the symmetric square $S^2(V_\lambda) = S^2(\mathbb{R}^{n(\lambda)+1})$ given by

$$W^0 = \text{span}_\mathbb{R}\{\rho(a)((v^0)^2) \in S^2(V_\lambda) | a \in G\} = \text{span}_\mathbb{R}\{(f_\lambda(x))^2 \in S^2(\mathbb{R}^{n(\lambda)+1}) | x \in G/K\}$$

and set

$$E = (W^0)^\perp \subset S^2(\mathbb{R}^{n(\lambda)+1}),$$

where the orthogonal complement is taken with respect to the scalar product $\langle A, B \rangle = \text{trace } B^t A$, $A, B \in S^2(\mathbb{R}^{n(\lambda)+1})$ ($t = \text{transpose}$). Finally, let $L \subset E$ be the convex body defined by

$$L = \{C \in E | C + I_{n(\lambda)+1} \text{ is positive semidefinite}\},$$

where $I_{n(\lambda)+1} = \text{identity of } \mathbb{R}^{n(\lambda)+1}$.

Given a full harmonic map $f: G/K \to S^n$ with $e(f) = \lambda/2$, the system $\{f^i\}_{i=0}^n$ (of components of $f$) is a linearly independent set in $V_\lambda$; in particular, $n \leq n(\lambda)$. By polar decomposition of matrices, there exists a positive semidefinite endomorphism $B \in S^2(\mathbb{R}^{n(\lambda)+1})$ such that $\text{im}(B \cdot f_\lambda) \subset S^{n(\lambda)}$ and $i \circ f$ is equivalent to $B \circ f_\lambda$, where $i: S^n \to S^{n(\lambda)}$ denotes the canonical inclusion map. Moreover, $B$ is uniquely determined by the equivalence class of $f$. Associate then to $f$ the matrix $C = B^2 - I_{n(\lambda)+1}$. As $B \circ f_\lambda$ maps into $S^{n(\lambda)}$, for $x \in G/K$, we have

$$\langle C, (f_\lambda(x))^2 \rangle = \langle C \cdot f_\lambda(x), f_\lambda(x) \rangle = \langle B \cdot f_\lambda(x), B \cdot f_\lambda(x) \rangle - 1 = 0$$

and hence $C \in L$. Now, by the same argument as in the proof of the Classification Theorem in [9], we obtain that the correspondence $f \to C$ gives rise to a parametrization of the space of equivalence classes of full harmonic maps $f: G/K \to S^n$ with $e(f) = \lambda/2$ by the convex body $L \subset E$. Moreover, $L$ is compact.

To prove the last statement of Theorem 1, let $f: G/K \to S^{n(\lambda)}$ be a full harmonic map with $e(f) = \lambda/2$. By Section 1, $v \in K(f)$ if and only if $\{v^i\}_{i=0}^{n(\lambda)} \subset V_\lambda$ with $\langle f, \delta \rangle = 0$. As $f$ is full there exists a unique $(n(\lambda) + 1) \times (n(\lambda) + 1)$ matrix $X$ such that $\delta = X \cdot f$, especially, $K(f) = PK(f)$. Writing $X = A + B$,
$A \in \text{so}(n(\lambda) + 1)$ and $B \in S^2(\mathbb{R}^{n(\lambda) + 1})$, the relation
\[
\langle \vec{f}, \theta \rangle = \langle \vec{f}, A \cdot \vec{f} \rangle + \langle \vec{f}, B \cdot \vec{f} \rangle = 0
\]
splits into $\langle \vec{f}, A \cdot \vec{f} \rangle = \langle \vec{f}, B \cdot \vec{f} \rangle = 0$. By fullness of $f$, there exists a non-singular $(n(\lambda) + 1) \times (n(\lambda) + 1)$ matrix $Y$ with $\vec{f} = Y \cdot \vec{f}_a$. Then, $Y^t \cdot B \cdot Y \in S^2(\mathbb{R}^{n(\lambda) + 1})$ and, for $x \in G/K$, we have
\[
O = \langle \vec{f}(x), B \cdot \vec{f}(x) \rangle = \langle Y \cdot \vec{f}_a(x), B \cdot Y \cdot \vec{f}_a(x) \rangle = \langle (Y^t \cdot B \cdot Y)(\vec{f}_a(x))^2, \rangle,
\]
i.e. $Y^t \cdot B \cdot Y \in E$. Now, the correspondence which associates to $v \in K(f)$ the pair $(A^o f, Y^t \cdot B \cdot Y) \in (\text{so}(n(\lambda) + 1))^o f \oplus E$ is a linear isomorphism. Hence $K(f)/\text{so}(n(\lambda) + 1)^o f \cong E$, which completes the proof of Theorem 1.

**Remark 6.** Setting
\[
W = \text{span}_\mathbb{R}(S^2((\vec{f}_a)_*T_\mathcal{A}(G/K))) \cap \{ x \in G/K \} \subset S^2(\mathbb{R}^{n(\lambda) + 1})
\]
and using the notations of Section 1, by [4], we have $W^0 \subset W$ and $L^0 = L \cap W^\perp$.

3. **Computation of $\dim L$ for spherical domains**

Let $G = \text{SO}(m + 1)$, $K = \text{SO}(m)$ and endow $G/K = S^m$ with the Euclidean metric. Then [1] Spec($S^m$) = \{ $\lambda_k = k(m + 1 - k)$ \} and, for each $k \in \mathbb{Z}_+$, the eigenspace $V_{\lambda_k}$ corresponding to $\lambda_k$ is the vector space $\mathcal{H}_m^k$ of spherical harmonics of order $k$ on $S^m$ with
\[
\dim \mathcal{H}_m^k = n(\lambda_k) + 1 = (2k + m - 1) \frac{(k + m - 2)!}{k!(m - 1)!}.
\]
Furthermore, as an orthogonal $\text{SO}(m + 1)$-module, $\mathcal{H}_m^k$ is irreducible [1] and, for fixed orthonormal base $\{ f_{\lambda_k}^i \}_{i = 1}^{n(\lambda_k)} \subset \mathcal{H}_m^k$, the construction of the standard minimal immersion $f_{\lambda_k}^i \colon S^m \to S^{n(\lambda_k)}$, in Section 2, shows that the $\text{SO}(m + 1)$-module structure $\rho$ on $\mathcal{H}_m^k$ is also class 1 for the pair $(\text{SO}(m + 1), \text{SO}(m))$ - i.e. there exists a unit vector $v^0 \in \mathcal{H}_m^k$ left fixed by $\rho(\text{SO}(m))$. (Here and in what follows we use the notions and results of [9] without making explicit references.) Conversely, every class 1 representation of $(\text{SO}(m + 1), \text{SO}(m))$ is equivalent to some $\mathcal{H}_m^k$ (considered as an irreducible orthogonal $\text{SO}(m + 1)$-module.) Denoting also by $\rho$ the induced representation on the symmetric square $S^2(\mathcal{H}_m^k)$, the $\text{SO}(m + 1)$-submodule
\[
W^0 = \text{span}_\mathbb{R}(\rho(a)((v^0)^2) \in S^2(\mathcal{H}_m^k) \mid a \in \text{SO}(m + 1))
\]
of $S^2(\mathcal{H}_m^k)$ is the sum of all submodules which are class 1 for $(\text{SO}(m+1), \text{SO}(m))$.

For $m = 2$, by elementary representation theory, each submodule of $S^2(\mathcal{H}_2^k)$ is class 1 for $(\text{SO}(3), \text{SO}(2))$ (since $\dim \mathcal{H}_2^k = 2k + 1$ is odd). Thus, $W^0 = S^2(\mathcal{H}_2^k)$ and hence $(W^0)^\perp = E = \{0\}$. For $k = 1$, a standard minimal immersion $f_{k_1}: S^m \to S^m$ is nothing but an isometry and hence infinitesimally rigid [8], i.e. by Theorem 1, $E = \{0\}$.

Setting $m > 2$ and $k > 1$, we now prove $(*)-(**)$. As irreducibility of submodules in $S^2(\mathcal{H}_m^k)$ do not depend on field extensions, the complexification $W^0 \otimes \mathbb{C}$ is the sum of all irreducible complex $\text{SO}(m+1)$-modules in $S^2(\mathcal{H}_m^k \otimes \mathbb{C})$ which are class 1 for $(\text{SO}(m+1), \text{SO}(m))$. By a result of Do Carmo–Wallach in [4] we have the following decompositions

\[
(*) \quad S^2(\mathcal{H}_3^k \otimes \mathbb{C}) = \mathcal{H}_3^{k/2} \mathcal{H}_3^{k-1} \otimes \mathbb{C}.
\]

and

\[
(**) \quad S^2(\mathcal{H}_m^k \otimes \mathbb{C}) = \mathcal{H}_m^{k/2} \mathcal{H}_m^{k-1} \otimes \mathbb{C}, \quad m > 3,
\]

where $V_\sigma^m$ stands for the irreducible complex $\text{SO}(m+1)$-module with highest weight $\sigma = (\sigma_1, \ldots, \sigma_l) \in (\mathbb{Z}/2\mathbb{Z})^l$, $l = [(m+1)/2]$. Since $V_{m}^{(l,0,\ldots,0)} = \mathcal{H}_m^l \otimes \mathbb{C}$, $i \in \mathbb{Z}_+$, $(*)$ and $(**)$ imply $(*)$ and $(**)$, resp.

REMARK 7. Once $\dim_c V_{m}^{(a,b,0,\ldots,0)}$ is known for each $(a,b) \in \Delta, a, b$ even, we can compute $\dim L = \dim E$ via $(*)-(**)$. In what follows, using the Weyl dimension formula [2] we determine $\dim_c V_{m}^{(a,b,0,\ldots,0)}$.

(i) $m = 2l$ even, $l > 2$. We have

\[
\dim_c V_{m}^{(a,b,0,\ldots,0)} = \frac{(a-b+1)(a+b+2l-2)}{2l-2} \times \prod_{r=3}^{l} \frac{(a+r-1)(a+2l-r)}{(r-1)(2l-r)}
\]

\[
\times \prod_{r=3}^{l} \frac{(b+r-2)(b+2l-r-1)}{(r-2)(2l-r-1)} \times \frac{(2a+2l-3)(2b+2l-3)}{(2l-1)(2l-3)}
\]

\[
= \frac{(a-b+1)(a+b+2l-2)}{2l-2} \times \frac{1}{a+1} \left( \begin{array}{c} a+2l-3 \\ a \\ b \\ b+2l-4 \end{array} \right)
\]

\[
\times \frac{(2a+2l-3)(2b+2l-3)}{(2l-1)(2l-3)}
\]

\[
= \frac{(a-b+1)(a+b+m-2)(2a+m-1)(2b+m-3)}{(a+1)(m-1)(m-2)(m-3)}
\]
(ii) $m = 2l - 1$ odd, $l > 2$. By similar computations we obtain the same formula for $\dim c V^{(a,b,0,\ldots,0)}_m$ as above.

(iii) $m = 3, 4$. Again by the Weyl dimension formula

$$\dim c V^{(a,b)}_3 = \dim c V^{(a,-b)}_3 = (a - b + 1)(a + b + 1)$$

and

$$\dim c V^{(a,b)}_4 = \frac{1}{6}(a - b + 1)(a + b + 2)(2a + 3)(2b + 1).$$

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Authors' addresses:

G. Tóth,
The Ohio State University,
Department of Mathematics,
231 West 18th Avenue,
Columbus, OH 43210
U.S.A.
(Received March 21, 1984)

G. D'Ambra,
Istituto Matematico,
Università di Cagliari,
Via Ospedale 72,
09100 Cagliari,
Italy