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TOROIDAL LIE GROUP ACTIONS ON COMPACT RIEMANNIAN MANIFOLDS AND THEIR RELATIONS TO THE FIBERING PROBLEM

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0. Introduction

In [3] A. Lichnerowicz proved that every compact oriented Riemannian manifold M with nonnegative Ricci tensor is the total space of a fibre bundle with flat torus base space and with totally geodesic bundle projection. He also showed that the universal covering $\tilde{M}$ splits isometrically as the product of a flat Euclidean space R^k and a compact Riemannian manifold M_0 with nonnegative Ricci tensor. Here

$$k = \max\{b_1(\hat{M})|\ \hat{M} \text{ is a finite covering of } M\}.$$

J. Cheeger and D. Gromoll obtained a stronger result [1], namely, they proved that there is a finite covering of M which is diffeomorphic with some $T^k \times M_0'$ but in many cases this splitting fails to be isometric.

The purpose of this paper is to generalize these results to the case where no curvature assumptions are supposed for M. More complete study of this topic can be found in [5]. Sections 1 and 2 contain nothing essentially new but the treatment slightly differs from the usual one as the basic ideas are accumulated in a commutative cube contained in Section 2. Our results are presented in Sections 3 and 4 together with some applications.

Throughout this paper all manifolds, maps, bundles, etc. will be smooth, i.e. of class C^∞, unless stated otherwise.

1. Preliminaries ([2] and [3])

Let M be a compact oriented Riemannian manifold with metric tensor $(\ ,\ )$. Suppose that this metric tensor is extended to an inner product of tensors of any type on M. If $\alpha, \beta \in \bigwedge^r(M)$ are r-forms on M and (α, β) denotes their inner product, then their *global scalar product* is defined by

$$\langle\alpha, \beta\rangle = \int_M (\alpha, \beta) v,$$

where $v \in \bigwedge^n(M)$, $n = \dim M$, is the volume element of M.

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The *operator of exterior codifferentiation* $\partial\colon \bigwedge^r(M) \to \bigwedge^{r+1}(M)$ is the adjoint of the exterior differentiation with respect to the global scalar product. The *Laplace operator* $\Delta\colon \bigwedge^r(M) \to \bigwedge^r(M)$ is defined by $\Delta = d \circ \partial + \partial \circ d$. It is an elliptic self-adjoint second-order differential operator and an r-form α is harmonic, i.e., $\Delta\alpha = 0$ if and only if $d\alpha = 0$ and $\partial\alpha = 0$. The dimension of the linear space $\mathscr{H}$ of harmonic 1-forms on M is the *first Betti number* $b_1(M) = p$ of M.

Denote by G the maximal connected subgroup of the group of isometries of M with respect to the compact-open topology. Then G is a compact Lie group and its Lie algebra L can be identified with the Lie algebra of Killing vector fields on M.

The isometries of M pull back harmonic 1-forms into harmonic 1-forms and thus $L_X\beta = 0$ holds for every $X \in L$ and $\beta \in \mathscr{H}$. Using the formula $L_X = d \circ i_X + i_X \circ d$, it follows that $i_X\beta$ is constant on M. Define

$$I = \{X \in L \mid i_X\beta = 0 \text{ for every } \beta \in \mathscr{H}\}.$$

If $X, Y \in L$ and $\beta \in \mathscr{H}$, then

$$0 = (d\beta)(X, Y) = \tfrac{1}{2}\big(X(i_Y\beta) - Y(i_X\beta) - i_{[X,Y]}\beta\big) = -\tfrac{1}{2} i_{[X,Y]}\beta$$

and thus $[L, L] \subset I$. Hence $I \subset L$ is an ideal and L/I is commutative. By the definition of I, it follows that if $X \in L$ has a critical point on M then $X \in I$.

Now, let M and M' be compact oriented Riemannian manifolds with metric tensors $(\ ,\ )$ and $(\ ,\ )'$, respectively, and let $f\colon M \to M'$ be a map of class C^2. Denote by $T^{[r]} = \bigwedge^r(T^*(M))$ and $F = f^*(T(M'))$ the bundle of r-covectors on M and the pull-back of the tangent bundle $T(M')$ of M' via f, respectively. Set $\bigwedge^r(M, F) = \Gamma(F \otimes T^{[r]})$, the space of r-forms on M with values in the vector bundle F. Then the elements of $\bigwedge^0(M, F)$ are nothing but the vector fields along f and the tangent map f_* turns out to be a specific 1-form on M with values in F. The Levi–Civita connections of M and M' yield a connection $\bar\nabla$ on the vector bundle $F \otimes T^{[r]}$ and it is orthogonal with respect to the induced Riemannian metric $(\ ,\ )$ on the fibres of $F \otimes T^{[r]}$. Thus $F \otimes T^{[r]}$ becomes a Riemannian-connected bundle. There is a first order differential operator

$$\bar d\colon \textstyle\bigwedge^r(M, F) \to \bigwedge^{r+1}(M, F)$$

characterized by the identity

$$\bar d(u \otimes \beta) = (\bar\nabla u) \wedge \beta + u \otimes (d\beta),$$

where $u \in \Gamma(F)$ and $\beta \in \bigwedge^r(M)$. Note that we cannot expect that $\bar d^2 = 0$ except in the flat case.

The operator of exterior codifferentiation

$$\bar\partial\colon \textstyle\bigwedge^r(M, F) \to \bigwedge^{r-1}(M, F)$$

is the adjoint of $\bar d$ with respect to the global scalar product. The Laplace operator

$$\bar\Delta\colon \textstyle\bigwedge^r(M, F) \to \bigwedge^r(M, F)$$

is defined by $\Delta = d \circ \bar{\partial} + \bar{\partial} \circ d$. It is an elliptic self-adjoint second-order differential operator and an r-form ϕ on M with values in F is harmonic, i.e., $\Delta\phi = 0$, if and only if $d\phi = 0$ and $\bar{\partial}\phi = 0$. The map f: $M \to M$ is said to be *harmonic* if $\Delta f_* = 0$. Every harmonic map is necessarily of class C^∞. By local calculation it can be shown that $df_* = 0$ always holds, i.e., harmonicity of f is equivalent to $\bar{\partial} f_* = 0$. The physical interpretation of harmonic maps is strongly connected with the energy functional

$$E(f) = \frac{1}{2} \int_M (f_*, f_*) v,$$

namely, it turns out that f is harmonic if and only if it is an extremal of the energy functional. Harmonic maps appear in many different problems of differential geometry. If $\dim M = 1$, then harmonic maps are the closed geodesics of M'. If M and M' are Kähler manifolds then holomorphic maps of M into M' are harmonic with respect to any compatible metrics.

A more restricted class of maps is that of totally geodesic maps, i.e., the maps for which $\bar{\nabla} f_* = 0$. They can be characterized by the property that they map geodesics into geodesics linearly.

By developing a general Weitzenböck formula for Riemannian-connected bundles, it follows that every harmonic map from a Riemannian manifold with nonnegative Ricci tensor into a nonpositively curved Riemannian manifold is totally geodesic.

2. Harmonic maps into tori ([3])

Let M be a compact oriented Riemannian manifold with first Betti number $b_1(M) = p$. If $m_0 \in M$ is a base point, then the total space of the universal covering π: $\tilde{M} \to M$ can be considered as the space of homotopy classes of curves starting from the point m_0. Denote $\tilde{m}_0 \in \tilde{M}$ the class of null-homotopic loops.

If $\beta \in \mathscr{H}$, then $d(\pi^*\beta) = \pi^*(d\beta) = 0$. Since $\tilde{M}$ is simply connected, $\pi^*\beta = du$ holds for some scalar u on $\tilde{M}$ determined up to an additive constant. Let

$$\tilde{J}: \tilde{M} \to \mathscr{H}^*$$

be defined by

$$\tilde{J}(\tilde{m})[\beta] = u(\tilde{m}) - u(\tilde{m}_0),$$

where $\pi^*\beta = du$. By the de Rham isomorphism, $H_1(M; R) \cong \mathscr{H}^*$. The first integral homology group $H_1(M; Z)$ maps onto a discrete subgroup of $H_1(M; R)$ of maximal rank. Denote by $P \subset \mathscr{H}^*$ the discrete subgroup corresponding to $H_1(M; Z)$ under the de Rham isomorphism.

The fundamental group $\pi_1(M, m_0)$ acts on $\tilde{M}$ such that the orbits are nothing but the fibres of π: $\tilde{M} \to M$. If $\tilde{m} \in \tilde{M}$, $s \in \pi_1(M, m_0)$ and $\beta \in \mathscr{H}$ with $\pi^*\beta = du$, then

$$(\tilde{J}(s\tilde{m}) - \tilde{J}(\tilde{m}))[\beta] = u(s\tilde{m}) - u(\tilde{m}) = \int_{\tilde{m}^{-1} \cdot s\tilde{m}} \beta,$$

where the last equality is obtained by the Stokes formula on M. Since $\tilde{m}^{-1} \cdot s\tilde{m}$ defines a 1-cycle, we have that $\tilde{J}(s\tilde{m}) - \tilde{J}(\tilde{m}) \in P$, i.e., the map $\tilde{J}$ can be projected down yielding a map $J\colon\ M \to \mathscr{H}^*/P$. The torus $B(M) = \mathscr{H}^*/P$ is called the *canonical torus of M* and thus we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{J}} & \mathscr{H}^* \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{J} & B(M) \end{array}$$

The components of J with respect to a flat coordinate neighbourhood of $B(M)$ can be expressed by the u's. Since $0 = \pi^*(\partial\beta) = \partial(\pi^*\beta) = \partial du = \Delta u$, these components of J are harmonic functions on M. Thus, if we endow $\mathscr{H}^*$ and $B(M)$ with the flat metric, we obtain that $J\colon\ M \to B(M)$ is harmonic.

The map J is called the *Jacobian map* of the Riemannian manifold M.

Let M and M' be compact oriented Riemannian manifolds with base points m_0 and m_0', respectively. Assume that $f\colon\ M \to M'$ is a base point preserving map such that f pulls back harmonic 1-forms of M' into harmonic 1-forms of M. Then f induces a linear map $f^*\colon \mathscr{H}' \to \mathscr{H}$. Since f is a base point preserving, there is a map $\tilde{f}\colon\ \tilde{M} \to \tilde{M}'$ such that the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M}' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

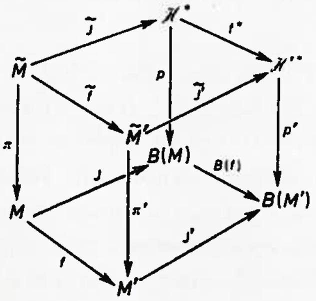
commutes. Denoting the dual map of $f^*\colon\ \mathscr{H}' \to \mathscr{H}$ by the same symbol, we have the following diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M}' \\ {\scriptstyle\tilde{J}}\downarrow & & \downarrow{\scriptstyle\tilde{J}'} \\ \mathscr{H}^* & \xrightarrow{f^*} & \mathscr{H}'^* \end{array}$$

In order to show that this diagram commutes, let $\tilde{m} \in \tilde{M}$, $\beta' \in \mathscr{H}'$ and $\pi'^*\beta' = du'$. Then $\pi^*f^*\beta' = \tilde{f}^*\pi'^*\beta' = \tilde{f}^*du' = d(u' \circ \tilde{f})$ and so $(f^* \circ \tilde{J})(\tilde{m})[\beta'] = \tilde{J}(\tilde{m})[f^*\beta']$ $= (u' \circ \tilde{f})(\tilde{m}) - (u' \circ \tilde{f})(\tilde{m}_0) = u'(\tilde{f}(\tilde{m})) - u'(\tilde{m}_0') = \tilde{J}'(\tilde{f}(\tilde{m}))[\beta']$ which accomplishes the proof. Under the de Rham isomorphisms $H_1(M; R) \cong \mathscr{H}^*$ and $H_1(M', R)$ $\cong \mathscr{H}'^*$ the induced homomorphisms correspond to each other and thus $f^*(P) \subset P'$. Thus the linear map f^* projects down to an affine map $B(f)\colon\ B(M) \to B(M')$ yielding the commutative diagram

$$\begin{array}{ccc} \mathscr{H}^* & \xrightarrow{f^*} & \mathscr{H}'^* \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B(M) & \xrightarrow{B(f)} & B(M') \end{array}$$

Each face of the cube



commutes and so the bottom face

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J'} \\ B(M) & \xrightarrow{B(f)} & B(M') \end{array}$$

also commutes.

In the special case where $M = M'$ and $f\colon M \to M'$ is an isometry we obtain a translation $B(f)\colon B(M) \to B(M)$. This leads to the following:

THEOREM 1. *The Jacobian map $J\colon M \to B(M)$ defines a homomorphism*

$$\hat{J}\colon G \to G_B,$$

where G_B is the group of translations of $B(M)$, such that $J \circ g = \hat{J}(g) \circ J$ holds, $g \in G$.

The Lie algebra L_B of G_B can be identified with the Lie algebra of uniform vector fields on $B(M)$. By local calculation it can be shown that every Killing vector field X on M projects down to a uniform vector field $J_*(X)$ on $B(M)$ and $J(\exp(tX)(m)) = \exp(tJ_*(X))(J(m))$ holds for every $t \in R$ and $m \in M$ and, moreover, $J_*(X) = 0$ if and only if $X \in I$. It follows that the homomorphism $\hat{J}$ induces J_* on the Lie algebra level and that the Lie algebra of $\Gamma = \ker \hat{J}$ is nothing but the ideal I.

Another application of the commutative cube can be obtained by assuming that $f\colon M \to M'$ is harmonic and that the Ricci tensor of M' is nonnegative. It is known that in this case $\mathcal{H}'$ consists of parallel forms. An easy local calculation shows that a harmonic map pulls back parallel 1-forms into harmonic 1-forms. Thus in this case the commutativity of the cube holds.

Quite specifically, let $M' = \theta$ be a flat torus. Then $B(\theta) = \theta$ and $J' = \mathrm{id}_\theta$ and hence we obtain the following:

THEOREM 2. *Every harmonic map $f\colon M \to \theta$ into a flat torus θ can be factorized through the Jacobian map yielding an affine map*

$$B(f)\colon B(M) \to \theta$$

such that $f = B(f) \circ J$.

It is not true in general that a harmonic map onto a positively curved manifold pulls back harmonic r-forms into harmonic r-forms, when $r \geqslant 2$, as the following example shows:

EXAMPLE. Let f: $T^2 \to S^2$ be a harmonic surjective map, [4]. By Sard's theorem there exists a regular point $x \in T^2$ of f, i.e., $(\operatorname{rank} f)(x) = 2$. If w denotes the volume element of S^2, then $(f^*w)(x) \neq 0$. Let us suppose that f^*w is harmonic, i.e., $f^*w = \lambda \cdot v$ for some $0 \neq \lambda \in R$, where v denotes the volume element of T^2. It means that $\operatorname{rank} f = 2$ everywhere on T^2, i.e., f is a local diffeomorphism. If g denotes the metric tensor of S^2, then f^*g is a metric tensor of T^2 and T^2 is nonnegatively curved with respect to this metric. Thus f^*g must be flat, i.e. S^2 can be endowed with a flat metric which is impossible. Hence f^*w is not harmonic.

Note that the preservation of harmonicity under harmonic mappings was extensively studied in [6].

3. Fibrations by the Jacobian map

In this section we shall investigate the following problem:

PROBLEM. Does the Jacobian map J: $M \to B(M)$ define a fibration?

The answer is "may be" and our main purpose is to give a sufficient condition for the affirmative answer. Our main result is the following:

THEOREM 3. *Let M be a compact oriented Riemannian manifold with Jacobian map J: $M \to B(M)$ and assume that* $\operatorname{rank} J \leqslant \operatorname{codim} I$ *everywhere on M. Then* $\operatorname{im} J = \vartheta \subset B(M)$ *is a toroid and*

$$J: M \to \vartheta$$

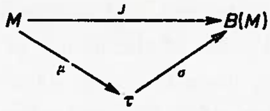
defines a harmonic fibre bundle with compact connected fibres and with finite commutative structure group.

Sketch proof. The ideal $I \subset L$ is a direct summand and thus we can choose a complementary ideal $P \subset L$ such that P is the Lie algebra of a closed and connected subgroup $Q \subset G$. Then $Q \cdot \Gamma = G$ and $Q \cap \Gamma = H \subset G$ is finite. Since $L' \subset I$, it follows that $Q \subset G$ is central and especially a toroidal subgroup of G. The subgroup $\operatorname{im} \hat{J} = H_B \subset G_B$ is a toroid in G_B and $\check{J} = \hat{J}|Q$: $Q \to H_B$ is a local isomorphism with kernel H. Then J is equivariant with respect to the local isomorphism $\check{J}$. If $m \in M$ and $y \in B(M)$, then denote by $\theta(m)$ and $\vartheta(y)$ the orbit of the action Q and H_B through m and y, respectively. Then, for every $m \in M$, the map

$$J|Q(m): Q(m) \to \vartheta(J(m))$$

is a finite covering and every isotropy subgroup of the action of Q on M is contained in H. Especially, $\operatorname{rank} J = \operatorname{codim} I$ everywhere on M. The image of J consists of a unique orbit ϑ of the action of H_B on $B(M)$. Thus $\vartheta \subset B(M)$ is a toroid and J: $M \to \vartheta$ is a harmonic map of maximal rank. If $y \in \vartheta$, then $J^{-1}(y) \subset M$ is a closed submanifold and if $h \in H_B$ and $g \in Q$ with $\hat{J}(g) = h$, then g maps $J^{-1}(y)$ isometrically onto $J^{-1}(h(y))$. Thus J: $M \to \vartheta$ is a fibre bundle with structure group H. Let $\sim$ be the relation on M defined by:

$m \sim m'$ if the points m and m' lie in the same connected component of a fibre. Then $\tau = M/\sim$ is a finite covering of ϑ and there is a factorization



with harmonic μ: $M \to \tau$. By Theorem 2 there exists an affine map ν: $B(M) \to \tau$ with $\mu \doteq \nu \circ J$. So, $\sigma \circ \nu \circ J = \sigma \circ \mu = J$, i.e., σ and ν are inverses of each other. Hence $\tau = B(M)$ and the fibres are connected which accomplishes the proof.

If the Ricci tensor field of M is nonnegative, then $\operatorname{rank} J = b_1(M) = \operatorname{codim} I$ and we obtain the classical result of [3] as follows:

COROLLARY. *Let M be a compact oriented Riemannian manifold with nonnegative Ricci tensor field. Then the Jacobian map J: $M \to B(M)$ defines a totally geodesic fibre bundle with compact connected fibres and with discrete (finite) commutative structure group.*

Remark. A similar statement is valid for compact Kähler manifolds. Let W be a compact Kähler manifold with $b_{1,0}(W) = p$. The maximal connected subgroup G of holomorphic transformations of W with respect to the compact-open topology is a complex Lie group and its Lie algebra L can be identified with the complex Lie algebra of infinitesimal holomorphic transformations of W. If H denotes the complex vector space of holomorphic 1-forms of type $(1, 0)$ on W, then let $I = \{A \in L |\ i_A \beta = 0$ for every closed $\beta \in H\}$. I has similar properties as that of the Riemannian case. An analogous construction yields the so-called Albanese map

$$J\colon W \to A(W),$$

where the Albanese torus $A(W)$ is of complex dimension p.

THEOREM 4. *Let W be a compact Kähler manifold with Albanese map J: $W \to A(W)$. Assume that:*

(i) *There exists a subalgebra in L complementary to I,*

(ii) $\operatorname{rank} J \leqslant \operatorname{codim} I$ *everywhere on W.*

Then there exist a complex torus ϑ with $\dim \vartheta = \operatorname{codim} I$ *and a holomorphic fibration J: $W \to \vartheta$ with compact connected fibres and with discrete structure group.*

Note that condition (i) is essential as one can see from the structure of complex solvable Lie algebras.

4. Coverings

The fibre structure of M defined in Theorem 3 gives some information about the structure of finite coverings of M.

THEOREM 5. *Let M be a compact oriented Riemannian manifold and assume that* $\operatorname{rank} J \leqslant \operatorname{codim} I = q$ *everywhere on M. Then there exists a finite covering $\varrho\colon \bar{M} \to M$ the total space of which splits diffeomorphically as $T^q \times M_0$, where M_0 is a closed submanifold of M. Especially, $\tilde{M}$ is diffeomorphic with the product $R^q \times \tilde{M}_0$.*

Sketch proof. We use the notations and terminology of the proof of Theorem 3. Let $\theta = \theta(m)$ be a principal orbit of the action of Q on M through some point $m_0 \in M$. Then $J|\theta\colon \theta \to \vartheta$ is a finite covering. Denote by $\bar{M}$ the pull-back of the bundle $J\colon M \to \vartheta$ via $J|\theta$. Then $\bar{M} = \{(m, m') \in \theta \times M |\ J(m) = J(m')\}$ and there is a commutative diagram

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\ \varrho\ } & M \\ \downarrow & & \downarrow J \\ \theta & \xrightarrow{\ J|\theta\ } & \vartheta \end{array}$$

Since $\varrho\colon \bar{M} \to M$ is a finite covering, it remains only to show that $\bar{M}$ has the required product structure. Let $\lambda\colon \bar{M} \to \theta \times M_0$, $M_0 = J^{-1}(J(m_0))$, be defined by $\lambda(m, m') = (m, g(m'))$, where $g \in Q$ such that $g(m) = g(m_0)$. Since θ is principal, it follows that λ is well-defined and diffeomorphism.

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