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# On variation spaces of harmonic maps into spheres

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## 1. Introduction

Given a harmonic map $f: M \to S^n$ [3] of a compact oriented Riemannian manifold $M$ into the Euclidean $n$-sphere $S^n$, $n \geqq 2$, a vector field $v$ along $f$, i.e. a section of the pull-back bundle $\mathscr{F} = f^*(T(S^n))$, gives rise to a (one-parameter, geodesic) variation $f_t = \exp\circ(tv): M \to S^n$, $t \in \mathbf{R}$, where $\exp: T(S^n) \to S^n$ is the exponential map. The element $v \in C^\infty(\mathscr{F})$ is said to be a *harmonic variation* if $f_t$ is harmonic for all $t \in \mathbf{R}$ and the set of all harmonic variations $v$ (or the variation space) of $f$ is denoted by $V(f) \subset C^\infty(\mathscr{F})$. Then [11] $v \in V(f)$ if and only if $\|v\| = \text{const.}$ and

(i) $\nabla^2 v = \text{trace}\, R(f_*, v) f_*$ (i.e. $v$ is a Jacobi field along $f$ [3]),

(ii) $\text{trace}\, \langle f_*, \nabla v \rangle = 0$,

where $\langle\, ,\rangle$ and $\nabla$ are the induced metric and connection of the Riemannian-connected bundle $\mathscr{F} \otimes \Lambda^*(T^*(M))$, $\nabla^2 = \text{trace}\, \nabla \circ \nabla$ [9], $R$ is the curvature tensor of $S^n$ and the differential $f_*$ of $f$ is considered as a section of $\mathscr{F} \otimes T^*(M)$. Denote by $K(f)$ the linear space of all vector fields $v$ along $f$ satisfying (i) and (ii). The equation (i) being (strongly) elliptic [9] $\dim K(f) < \infty$ and $V(f) = \{v \in K(f) \,|\, \|v\| = \text{const.}\} \subset K(f)$ is a subset with the obvious property $\mathbf{R} V_0(f) = V(f)$, where $V_0(f) = \{v \in K(f) \,|\, \|v\| = 1\}$.

The purpose of this paper is to give a geometric description of the variation space $V(i) \subset K(i)$ $(\cong \mathbf{R}^N)$ of the canonical inclusion $i: S^m \to S^n$, where $N = m(m+1)/2 + (n-m)(m+1)$. In Section 2 we collect the necessary tools from matrix theory used in the sequel, especially we describe the singular value decomposition of rectangular matrices (see e.g. [7]). In Section 3 the problem of determining $V_0(i)$ is reduced to the geometric characterization of an (algebraic) set of matrices. Then the singular value decomposition of these matrices are exploited to get a description of $V_0(i) \subset K(i)$ as a set of orbits (under a linear Lie group action) which contains a (twisted) simplex as a global section (Theorem 1). In particular, we prove that

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$V(\mathrm{id}_{S^{2r-1}})$, $r \in \mathbf{N}$, is the double cone over the irreducible Hermitian symmetı space $SO(2r)/U(r)$ $(=V_0(\mathrm{id}_{S^{2r-1}}))$. (Note that $V(\mathrm{id}_{S^{2r}})=0$ because $\chi(S^{2r})=2$ [11
In Section 4 we first give an alternative description of the linear space $K(f)$. particular, we obtain that there is a one-to-one correspondence between the elemen of $V_0(f)$ and the orthogonal pairs $f, f^1 : M \to S^n$ of harmonic maps with the san energy density $e(f)=e(f^1)$ [3]. Second, as an example, we determine $K(f)$ for tl Veronese surface $f : S^2 \to S^4$ and prove that $K(f) \cong K(\mathrm{id}_{S^4})$ and $V(f)=V(\mathrm{id}_{S^4})=$ hold.

Throughout this paper all manifolds, maps, bundles, etc. will be smooth, i. of class $C^\infty$. The report [3] is our general reference for harmonic maps though v adopt the sign conventions of [6].

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## 2. Preliminaries from matrix theory

First we fix some notations used in the sequel. Denote by $M(p,q)$ the lin space of $(p \times q)$ matrices and, as usual, let $I_p$ and $0_p$ the unit and zero eleme of $M(p,p)$. A matrix $A \in M(p,q)$ with entries $a_{ij}$, $i=1,\dots,p$, $j=1,\dots,q$, said to be (rectangular) diagonal if

$$a_{ij} = \begin{cases} 0, & \text{if } i \neq j,\ i=1,\dots,p,\ j=1,\dots,q, \\ \sigma_i, & \text{if } i=j,\ i=1,\dots,\min(p,q) \end{cases}$$

holds. We write $A=\mathrm{diag}(\sigma_1,\dots,\sigma_d)^p_q$ with $d=\min(p,q)$ and, in case $p$ we omit the indices $p$ and $q$.

The singular value decomposition of rectangular matrices is given in the fo ing theorem. (For the proof, see [7].)

Theorem A. *For any matrix $B \in M(p,q)$ there exist orthogonal ma $V \in O(p)$ and $U \in O(q)$ such that*

$$V^{\mathrm{T}} B U = \mathrm{diag}(\sigma_1,\dots,\sigma_d)^p_q$$

*with $\sigma_i \geqq 0$, $i=1,\dots,d=\min(p,q)$. The matrices $V, U$ and the values σ determined by the relations:*

(A₁) $V^{\mathrm{T}} B B^{\mathrm{T}} V = \mathrm{diag}(\sigma_1^2,\dots,\sigma_d^2,\dots,\sigma_p^2)$,

(A₂) $U^{\mathrm{T}} B^{\mathrm{T}} B U = \mathrm{diag}(\sigma_1^2,\dots,\sigma_d^2,\dots,\sigma_q^2)$,

(A₃) $BU = V\,\mathrm{diag}(\sigma_1,\dots,\sigma_d)^p_d$,

*where $\sigma_i=0$ for $d<i\leqq\max(p,q)$.*

Remark. The numbers $\sigma_i \geqq 0$; $i=1, \dots, d$, are called the singular values of $B$. Clearly, $V$ and $U$ can always be chosen such that $\sigma_1 \geqq \sigma_2 \geqq \dots \geqq \sigma_d$ holds.

Denote by $\Lambda_r \in so(2r)$ the skew-symmetric matrix

$$\Lambda_r = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$$

and put $\Lambda_1 = \Lambda$. In the next theorem we collect some properties of skew-symmetric matrices (cf. [8] pp. 151, 231).

Theorem B. *For any matrix* $\mathcal{J} \in so(p)$ *we have*

(B$_1$) rank $\mathcal{J} = 2r \leqq p$;

(B$_2$) *The* $2r$ *nonzero eigenvalues of* $\mathcal{J}$ *appear in pairs* $\lambda_{2i-1} = \lambda_{2i} = \pm\sqrt{-1}\sigma_i$ *with* $\sigma_i > 0$, $i=1, \dots, r$, *while zero is an eigenvalue with multiplicity* $p-2r$;

(B$_3$) *There exists* $U \in O(p)$ *such that*

(1) $$U^{\mathrm{T}} \mathcal{J} U = \operatorname{diag}(0_{p-2r}, \sigma_1 \Lambda, \dots, \sigma_r \Lambda)$$

*or equivalently*

(1') $$U^{\mathrm{T}} \mathcal{J} U = \begin{cases} \operatorname{diag}(\hat{\sigma}_1 \Lambda, \dots, \hat{\sigma}_{p/2} \Lambda), & \text{if } p \text{ is even}, \\ \operatorname{diag}(0, \hat{\sigma}_1 \Lambda, \dots, \hat{\sigma}_{[p/2]} \Lambda), & \text{if } p \text{ is odd}, \end{cases}$$

*where* $\hat{\sigma}_1 = \dots = \hat{\sigma}_{[(p-2r)/2]} = 0$ *and* $\hat{\sigma}_{[(p-2r)/2]+i} = \sigma_i$, $i = 1, \dots, r$;

(B$_4$) *With the same matrix* $U \in O(p)$ *we have*

(2) $$U^{\mathrm{T}}(-\mathcal{J}^2) U = \begin{cases} \operatorname{diag}(\hat{\sigma}_1^2 I_2, \dots, \hat{\sigma}_{p/2}^2 I_2) & \text{if } p \text{ is even}, \\ \operatorname{diag}(0, \hat{\sigma}_1^2 I_2, \dots, \hat{\sigma}_{[p/2]}^2 I_2), & \text{if } p \text{ is odd}, \end{cases}$$

*In particular, the nonzero singular values of* $\mathcal{J}$ *have even multiplicities.*

## 3. Variation space of the canonical inclusion $i: S^m \to S^n$

Let $i: S^m \to S^n$ be the canonical inclusion and let $W^1, \dots, W^k$, $k=n-m$, denote the system of orthonormal parallel sections of the normal bundle of $i$ defined by the standard base vectors $e_{m+2}, \dots, e_{n+1} \in \mathbf{R}^{n+1}$.

According to a result of [11] $v \in K(i)$ if and only if the tangential part $\mathcal{J}$ of $v$ is a Killing vector field on $S^m$ and there exist vectors $b_1, \dots, b_k \in \mathbf{R}^{m+1}$ such that the orthogonal decomposition

$$v_x = \mathcal{J}_x + \sum_{j=1}^{k} \langle b_j, x \rangle W_x^j, \quad x \in S^m,$$

is valid. Hence the linear map $\Psi: K(i) \to so(m+1) \times M(k, m+1)$ defined by $\Psi(v) = (\mathcal{J}, B)$, $v \in K(i)$, where $\mathcal{J}$ is the tangential part of $v$ and $B \in M(k, m+1)$

consists of the row vectors $b_1, \dots, b_k \in \mathbf{R}^{m+1}$ occurring in the decompositio $v$ above, is a linear isomorphism. In what follows, we identify $K(i)$ $so(m+1)\times M(k, m+1)$ via $\Psi$. Further, $V(i)=\mathbf{R}V_0(i)\subset K(i)$, where $V_0$ $=\{v\in K(i) \mid \|v\|=1\}$. Thus, for $v=(\mathcal{J}, B)\in V_0(i)$, we have

$$1 = \|v_x\|^2 = \|\mathcal{J}_x\|^2 + \sum_{j=1}^{k} \langle b_j, x\rangle^2 = \langle -\mathcal{J}^2 x, x\rangle + \langle B^{\mathrm{T}} B x, x\rangle, \quad x\in S^m,$$

i.e.

$$V_0(i) = \{(\mathcal{J}, B)\in so(m+1)\times M(k, m+1) \mid -\mathcal{J}^2 + B^{\mathrm{T}}B = I_{m+1}\}.$$

The objective of this section is to give a geometric description of the set $V_0$ $\subset K(i)$. Before stating our main theorem we introduce some notations. For the g positive integers $m$ and $n$, $m\leqq n$, set

$$\iota = \begin{cases} \min\left((m+1)/2, [k/2]\right), & \text{if } m+1 \text{ is even,} \\ \min\left(m/2, [(k-1)/2]\right), & \text{if } m+1 \text{ is odd,} \end{cases}$$

where $k=n-m$, and define

$$\Delta_\iota = \{(\sigma_1, \dots, \sigma_\iota)\in \mathbf{R}^\iota \mid 1 \geqq \sigma_1 \geqq \dots \geqq \sigma_\iota \geqq 0\}.$$

So $\Delta_\iota\subset \mathbf{R}^\iota$ is a (linear) simplex which reduces to a point if $\iota=0$. (Note that $\iota \geqq$ and equality holds if and only if $m=n$ is even, in which case $V_0(i)=\emptyset$ [11] an put $\Delta_{-1}=\emptyset$.)

A linear representation of the Lie group $O(m+1)\times O(k)$ on the vector s $K(i)=so(m+1)\times M(k, m+1)$ is given by

$$(U, V)\cdot(\mathcal{J}, B) = (U\mathcal{J}U^{\mathrm{T}}, VBU^{\mathrm{T}}),$$

$(U, V)\in O(m+1)\times O(k)$, $(\mathcal{J}, B)\in so(m+1)\times M(k, m+1)$. Clearly, the subset $V_0($ $\subset K(i)$ is invariant, i.e. $V_0(i)$ is the union of orbits crossing $V_0(i)$. Finally introduce certain subgroups of $O(m+1)\times O(k)$ which will be the isotropy groups at points of $V_0(i)$. For given nonnegative integers $a_0, b_0, c_1, c_2, \dots,$ with $m+1=a_0+2c_1+\dots+2c_{s+1}$ and $k=a_0+2c_1+\dots+2c_s+b_0$ define the subgrc

$$\mathcal{G}(c_1, \dots, c_{s+1}) = \{(A_0, C_1, \dots, C_{s+1}; A_0, C_1, \dots, C_s, B_0)\in O(m+1)\times O(k) \mid$$
$$A_0\in O(a_0),\quad B_0\in O(b_0), C_i\in U(c_i),\quad i=1, \dots, s+1\},$$

where $U(c_i)$ is considered as a subgroup of $SO(2c_i)$ via the canonical embed $U(c_i)\to SO(2c_i)$, $i=1, \dots, s+1$. The isotropy type i.e. the set of all conjugacy cla of a subgroup $\mathcal{G}\subset O(m+1)\times O(k)$ is denoted by $(\mathcal{G})$. The main result of section is the following:

Theorem 1. *There exists an embedding* $\Phi: \Delta_\iota\to K(i)$ *such that* $\Phi(\Delta_\iota)$ *a global section of the invariant subset* $V_0(i)$ *(i.e.* $\Phi(\Delta_\iota)\subset V_0(i)$ *and any orbi*

$(i)$ *cuts* $\Phi(\Delta_t)$ *at exactly one point). Moreover, for* $\sigma=(\sigma_0, \ldots, \sigma_0, \sigma_1, \ldots, \sigma_1, \ldots,$ $\sigma_{s+1}, \ldots, \sigma_{s+1})\in\Delta_t$, *where* $1=\sigma_0>\sigma_1>\ldots>\sigma_s>\sigma_{s+1}=0$ *and* $\sigma_i$ *occurs* $c_i$ *times in* $\sigma$, $i=0, \ldots, s+1$, *the isotropy type of the orbit through* $\Phi(\sigma)$ *is* $(\mathscr{G}(c_1, \ldots, c_s, c_{s+1}+([(m+1)/2]-t)^+)$ ($^+=$ *positive part*) *or equivalently this orbit has the form*

$$(O(m+1)\times O(k))/\mathscr{G}(c_1, \ldots, c_s, c_{s+1}+([m+1)/2]]-t)^+).$$

*In particular, for each open face* $\Delta$ *of the simplex* $\Delta_t$ *the orbits through* $\Phi(\Delta)$ *have the same type.*

Remarks 1. Each orbit consists of 1, 2 or 4 components. More precisely, the subgroups $\mathscr{G}(c_1, \ldots, c_{s+1})\subset SO(m+1)\times SO(k)$ being connected, the orbit $O(m+1)\times O(k))/\mathscr{G}(c_1, \ldots, c_s, c_{s+1}+([(m+1)/2]-t)^+)$ has $N$ components, where

$$N=\begin{cases}1, & \text{if } k>0 \text{ and } a_0b_0>0,\\ 2, & \text{if } k>0,\ a_0b_0=0 \text{ and } a_0+b_0>0 \text{ or if } k=0,\\ 4, & \text{if } k>0 \text{ and } a_0=b_0=0.\end{cases}$$

2. By a result of [13] for any locally rigid harmonic embedding $f: M\to S^n$ we have $V(f)=V(i)$, where $i: S^m\to S^n$ is the inclusion and $m$ is the dimension of the least totally geodesic submanifold of $S^n$ containing the image of $f$. Thus Theorem 1 gives a description of the variation space of all locally rigid harmonic embeddings.

The proof of Theorem 1 is broken up into a few lemmas. Let $(\mathscr{J}, B)\in V_0(i)$ be fixed. Then, by Theorem B, there exists $U\in O(m+1)$ such that $U^{\mathrm{T}}\mathscr{J}U$ and $U^{\mathrm{T}}(-\mathscr{J}^2)U$ have the form (1′) and (2), resp., with

$$0\leqq \vartheta_1\leqq\ldots\leqq\vartheta_{[(m+1)/2]}.$$

Thus, by $B^{\mathrm{T}}B=I_{m+1}+\mathscr{J}^2$, we obtain

$$U^{\mathrm{T}}B^{\mathrm{T}}BU=\begin{cases}\operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{[(m+1)/2]}^2 I_2), & \text{if } m+1 \text{ is even},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{[(m+1)/2]}^2 I_2), & \text{if } m+1 \text{ is odd},\end{cases}$$

where $\sigma_i^2=1-\vartheta_i^2$, $i=1, \ldots, [(m+1)/2]$. Clearly, $1\geqq\sigma_1^2\geqq\ldots\geqq\sigma_{[(m+1)/2]}^2\geqq 0$ is satisfied. Then the values $\sigma_i^2$, $i=1, \ldots, [(m+1)/2]$, occurring twice in $B^{\mathrm{T}}B$, are the eigenvalues of the positive semidefinite matrix $B^{\mathrm{T}}B$. The nonzero eigenvalues of $B^{\mathrm{T}}B$ and $BB^{\mathrm{T}}$ being the same, the system of eigenvalues of $BB^{\mathrm{T}}\in M(k,k)$ can be obtained from that of $B^{\mathrm{T}}B\in M(m+1, m+1)$ by supplementing or omitting $|k-(m+1)|$ zeros according as $k\geqq m+1$ or $k<m+1$. In the latter case, for some index $t_0\leqq[k/2]$, $\sigma_i=0$, $i>t_0$, must be valid. The determination of the minimal value of $t_0$ can be done by making distinction according to the parity of $k$. Hence

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we have

$$U^{\mathrm{T}}B^{\mathrm{T}}BU=$$

$$=\begin{cases}\operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{(m+1)/2}^2 I_2) & \text{for } k \geqq m+1,\ m+1 \text{ even},\\ \operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{k/2}^2 I_2, 0_{m+1-k}) & \text{for } k \text{ even},\ k < m+1,\ m+1 \text{ even},\\ \operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{[k/2]}^2 I_2, 0_{m+1-2[k/2]}) & \text{for } k \text{ odd},\ k < m+1,\ m+1 \text{ even},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{[(m+1)/2]}^2 I_2) & \text{for } k \geqq m+1,\ m+1 \text{ odd},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{(k-1)/2}^2 I_2, 0_{m+1-k}) & \text{for } k \text{ odd},\ k < m+1,\ m+1 \text{ odd},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{[(k-1)/2]}^2 I_2, 0_{m-2[(k-1)/2]}) & \text{for } k \text{ even},\ k < m+1,\ m+1 \text{ o}\end{cases}$$

A case-by-case verification shows that the minimal value of $t_0$ is the numb defined before Theorem 1. Thus we obtain

$$U^{\mathrm{T}}B^{\mathrm{T}}BU=\begin{cases}\operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_t^2 I_2, 0_{m+1-2t}), & \text{if } m+1 \text{ is even},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_t^2 I_2, 0_{m-2t}), & \text{if } m+1 \text{ is odd},\end{cases}$$

and consequently (1′) has the form

$$U^{\mathrm{T}}\mathscr{J}U=\begin{cases}\operatorname{diag}(\vartheta_1 \Lambda, \ldots, \vartheta_t \Lambda, \Lambda_{(m+1-2t)/2}), & \text{if } m+1 \text{ is even},\\ \operatorname{diag}(0, \vartheta_1 \Lambda, \ldots, \vartheta_t \Lambda, \Lambda_{(m-2t)/2}), & \text{if } m+1 \text{ is odd}.\end{cases}$$

Lemma 1. *Let $(\mathscr{J}, B)\in K(i)$. Then $(\mathscr{J}, B)\in V_0(i)$ if and only if there e* *$(U, V)\in O(m+1)\times O(k)$ such that $(\mathscr{J}, B)=(U\mathscr{J}(\vartheta)U^{\mathrm{T}}, VB(\sigma)U^{\mathrm{T}}$, where*

$$\mathscr{J}(\vartheta)=\begin{cases}\operatorname{diag}(\vartheta_1 \Lambda, \ldots, \vartheta_t \Lambda, \Lambda_{(m+1-2t)/2}) & \text{if } m+1, \text{ is even}\\ \operatorname{diag}(0, \vartheta_1 \Lambda, \ldots, \vartheta_t \Lambda, \Lambda_{(m-2t)/2}), & \text{if } m+1, \text{ is odd},\end{cases}$$

$$B(\sigma)=\begin{cases}\operatorname{diag}(\sigma_1 I_2, \ldots, \sigma_t I_2, 0_{d-2t})_k^{m+1}, & \text{if } m+1, \text{ is even},\\ \operatorname{diag}(1, \sigma_1 I_2, \ldots, \sigma_t I_2, 0_{d-1-2t})_k^{m+1}, & \text{if } m+1, \text{ is odd},\end{cases}$$

*with $\sigma\in\Delta_t$, $\vartheta_i=\sqrt{1+\sigma_i^2}$, $i=1, \ldots, t$, and $d=\min(m+1, k)$.*

Proof. If $(\mathscr{J}, B)\in V_0(i)$ then there exists $U\in O(m+1)$ such that $U^{\mathrm{T}}B^{\mathrm{T}}$ $=B(\sigma)^{\mathrm{T}}B(\sigma)$ and $U^{\mathrm{T}}\mathscr{J}U=\mathscr{J}(\vartheta)$ with $0\leqq\vartheta_1\leqq\ldots\leqq\vartheta_{[(m+1)/2]}$. The diag entries of $U^{\mathrm{T}}B^{\mathrm{T}}BU$ are the eigenvalues of $B^{\mathrm{T}}B$ and hence, by Theorem A, exists $V\in O(k)$ such that the pair $(U, V)$ perform the singular value decompos of $B$, i.e. we have $V^{\mathrm{T}}BU=B(\sigma)$. Thus, $(U^{\mathrm{T}}\mathscr{J}U, V^{\mathrm{T}}BU)=(\mathscr{J}(\vartheta), B(\sigma))$, c The converse being obvious the proof is finished.

By the lemma above the map $\Phi: \Delta_t\to K(i)$, $\Phi(\sigma)=(\mathscr{J}(\vartheta), B(\sigma))$, $\sigma\in\Delta_t$, embedding with $(O(m+1)\times O(k))\cdot\Phi(\Delta_t)=V_0(i)$. Moreover, the eigenvalue $\mathscr{J}$ and the singular values of $B$ are invariants characterizing the orbit thr $(\mathscr{J}, B)$ uniquely. Thus $\Phi(\Delta_t)$ is a global section on $V_0(i)$ which accomplishe proof of the first statement of Theorem 1.

Let $\sigma=(\sigma_0,\dots,\sigma_0,\sigma_1,\dots,\sigma_1,\dots,\sigma_{s+1},\dots,\sigma_{s+1})\in\Delta_t$ be fixed with $1=\sigma_0>$ $\sigma_1>\dots>\sigma_s>\sigma_{s+1}=0$ and $\sigma_i$ occurs $c_i$ times in $\sigma$, $i=0,\dots,s+1$. It remains compute the isotropy type of the orbit through $\Phi(\sigma)$. The isotropy subgroup $\Phi(\sigma)$ consists of pairs $(U,V)$ such that $U\mathcal{J}(\vartheta)=\mathcal{J}(\vartheta)U$ and $VB(\sigma)=B(\sigma)U$. rst we study the second relation. Consider $B(\sigma)\in M(k,m+1)$ as a matrix

$$B(\sigma)=\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix},$$

here $\Sigma=\operatorname{diag}(\sigma_0 I_{a_0},\sigma_1 I_{2c_1},\dots,\sigma_s I_{2c_s})\in M(r,r)$, $r=a_0+2\sum_{i=1}^{s}c_i$,

$$a_0=\begin{cases}2c_0, & \text{if } m+1 \text{ is even},\\ 2c_0+1, & \text{if } m+1 \text{ is odd},\end{cases}$$

nd $0$ on the right lower corner is of size $(k-r)\times(m+1-r)$.

Lemma 2. *Let $(U,V)\in O(m+1)\times O(k)$ such that $VB(\sigma)=B(\sigma)U$ holds. hen we have $V=\operatorname{diag}(A_0,C_1,\dots,C_s,B_0)$ and $U=\operatorname{diag}(A_0,C_1,\dots,C_s,C_{s+1})$, here $A_0\in O(a_0)$, $B_0\in O(k-r)$, $C_i\in O(2c_i)$, $i=1,\dots,s$, $C_{s+1}\in O(m+1-r)$.*

Proof. Let $V\in O(k)$ and $U\in O(m+1)$ have the partitioned forms (conformal o that of $B(\sigma)$ above):

$$V=\begin{bmatrix}V_0 & R\\ S & B_0\end{bmatrix}\quad\text{and}\quad U=\begin{bmatrix}U_0 & P\\ Q & C_{s+1}\end{bmatrix},$$

where $V_0, U_0\in M(r,r)$, $B_0\in M(k-r,k-r)$, $C_{s+1}\in M(m+1-r,m+1-r)$. (The size of $C_{s+1}$ can be expressed as $m+1-r=2c_{s+1}+2([(m+1)/2]-t)^+$). Substituting these into the equations $VB(\sigma)=B(\sigma)U$, $VV^{\mathrm{T}}=I_k$, $UU^{\mathrm{T}}=I_{m+1}$ we obtain $R=0$, $S=0$, $V_0\in O(r)$, $B_0\in O(k-r)$ and $P=0$, $Q=0$, $U_0\in O(r)$, $C_{s+1}\in O(m+1-r)$. Thus the first equation reduces to $V_0\Sigma=\Sigma U_0$, i.e. by $\det\Sigma=\sigma_1^{2c_1}\dots\sigma_s^{2c_s}>0$, $V_0=\Sigma U_0\Sigma^{-1}$. Substituting this into the orthogonality relation $V_0^T V_0=I_r$ we get $U_0\Sigma^2=\Sigma^2 U_0$ which gives for $U_0=(C_{ij})$, $C_{00}\in M(a_0,a_0)$, $C_{i0}\in M(2c_i,a_0)$, $C_{0j}\in M(a_0,2c_j)$, $C_{ij}\in$ $\in M(2c_i,2c_j)$, $i,j=1,\dots,s$, the relations $C_{ij}=0$, if $i\neq j$. Hence, using the notations $C_{00}=A_0$ and $C_{ii}=C_i$, $i=1,\dots,s$, we obtain $U_0=\operatorname{diag}(A_0,C_1,\dots,C_s)$ with $A_0\in O(a_0)$, $C_i\in O(2c_i)$, $i=1,\dots,s$. As $U_0$ and $\Sigma$ commute we have $V_0=U_0$ which accomplishes the proof.

Consider now the second equation $U\mathcal{J}(\vartheta)=\mathcal{J}(\vartheta)U$, where $U$ has the form given in Lemma 2. Clearly, this equation is satisfied if and only if $C_i\in Z(\Lambda_{c_i})$, $i=1,\dots,s$, $C_{s+1}\in Z(\Lambda_{(m+1-r)/2})$, where $Z(\Lambda_p)$ denotes the centralizer of $\Lambda_p$ in $O(2p)$.

**Lemma 3.** *The centralizer* $Z(\Lambda_p) \subset O(2p)$ *is connected and there exi* $U_0 \in O(2p)$ *such that* $\operatorname{Ad}(U_0) Z(\Lambda_p) = U(p) \subset SO(2p)$, *where* Ad *denotes the adjo representation of* $O(2p)$.

**Proof.** It is well-known that $Z(\Lambda_p) \subset SO(2p)$ (cf. [8], Ch. IV. § 29, p. 248). F we prove that $Z(\Lambda_p) \subset SO(2p)$ is connected. Clearly, $\exp((\pi/2)\Lambda_p) = \Lambda_p$, wh $\exp: so(2p) \to SO(2p)$ is the exponential map. Hence $T = \overline{\exp(\mathbf{R}\Lambda_p)} \subset SO(2p)$ a toroidal subgroup which contains $\Lambda_p$, i.e. its centralizer $Z(T)$ is contained $Z(\Lambda_p)$. On the other hand, if $U \in Z(\Lambda_p)$ then the geodesics $s \mapsto \exp((\pi/2)s\Lambda_p) \cdot$ $s \mapsto U \cdot \exp((\pi/2)s\Lambda_p)$, $s \in \mathbf{R}$, (with respect to a biinvariant metric on $SO(2p)$) h common tangent vector at $s=0$, i.e. $\exp((\pi/2)s\Lambda_p)U = U\exp((\pi/2)s\Lambda_p)$ wh implies that $U \in Z(T)$. Thus $Z(\Lambda_p) = Z(T)$ and hence connected (cf. [4], C 2.8. p. 287). Finally, let

$$\mathcal{J}_p = \begin{bmatrix} 0_p & I_p \\ -I_p & 0_p \end{bmatrix}$$

and choose $U_0 \in O(2p)$ with $\operatorname{Ad}(U_0)\Lambda_p = \mathcal{J}_p$. Then $\operatorname{Ad}(U_0)Z(\Lambda_p) = Z(\operatorname{Ad}(U_0)\Lambda_p)$ $= Z(\mathcal{J}_p)$ and the fixed point set of the automorphism $\operatorname{Ad}(\mathcal{J}_p)$ of $SO(2p)$ is $Z($ It is known that $Z(\mathcal{J}_p) = U(p) \subset SO(2p)$ ([4], p. 453—454) which accomplishes proof.

By Lemmas 1—3, $(U, V)$ belongs to the isotropy subgroup at $\Phi(\sigma)$ if only if $(U, V) \in O(m+1) \times O(k)$ is conjugate to an element of $\mathcal{G}(c_1, \ldots, c_s, c_{s+}$ $+([(m+1)/2]-t)^+)$ (under a conjugation which does not depend on $(U,$ which completes the proof of Theorem 1.

**Example** (Variation space of the identity of odd spheres). Consider the spe case when $m=n=2r-1$ odd. Then $t=0$ and $V_0(\mathrm{id}_{S^{2r-1}})$ reduces to a si orbit through $\Lambda_r \in so(2r)$ under the adjoint representation of $O(2r)$ on $so$ We claim that this orbit is a disjoint union

$$\operatorname{Ad}(SO(2r))\Lambda_r \cup \operatorname{Ad}(SO(2r))\Lambda_r^-,$$

where $\Lambda_r^- = \operatorname{diag}(\Lambda, \ldots, \Lambda, -\Lambda) \in so(2r)$. Indeed, denoting $R = \operatorname{diag}(1, \ldots, 1, -$ $\in O(2r)$, we have $R\Lambda_r R = \Lambda_r^-$ and hence if $U \in O(2r)$ such that $\operatorname{Ad}(U)\Lambda_r =$ then $\operatorname{Ad}(RU)\Lambda_r = \Lambda_r$ which implies $RU \in SO(2r)$, i.e. $\det U = -1$.

The Killing form of $so(2r)$ is a negative definite Ad-invariant scalar pro on $so(2r)$ and so it follows easily that any ray in $so(2r)$ starting at the origin the orbit $\operatorname{Ad}(SO(2r))\Lambda_r$ (or $\operatorname{Ad}(SO(2r))\Lambda_r^-$) at most once.

*Case I: r is even.* Then $\operatorname{Ad}(U_0)\Lambda_r = -\Lambda_r$ with $U_0 = \operatorname{diag}(1, -1, 1, -1$ $\ldots, 1, -1) \in SO(2r)$, i.e. the orbit $\operatorname{Ad}(SO(2r))\Lambda_r$ (and $\operatorname{Ad}(SO(2r))\Lambda_r^-$ central symmetric to the origin. Thus $V(\mathrm{id}_{S^{2r-1}}) = \mathbf{R} \cdot V_0(\mathrm{id}_{S^{2r-1}})$ is a do cone over $\operatorname{Ad}(SO(2r))\Lambda_r = SO(2r)/U(r)$.

*Case II: $r$ is odd.* It follows easily that any line through the origin cuts $V_0(\mathrm{id}_{S^{2r-1}})$ twice and that the components $\mathrm{Ad}(SO(2r))\Lambda_r$ and $\mathrm{Ad}(SO(2r))\Lambda_r^-$ are central symmetric to each other, i.e. $V(\mathrm{id}_{S^{2r-1}})$ is again a double cone over $SO(2r)/U(r)$.

Remark. In the special case $r=2$ the space $V_0(\mathrm{id}_{S^3})$ is the disjoint union of two samples of $S^2(=SO(4)/U(2))$ which was already noticed in [13].

## 4. The Veronese surface

Let $M$ be a compact oriented Riemannian manifold and consider a harmonic map $f: M \to S^n$. By the inclusion $j: S^n \to \mathbf{R}^{n+1}$ the map $f$ becomes a vector-valued function $f: M \to \mathbf{R}^{n+1}$. Moreover, translating vectors tangent to $S^n \subset \mathbf{R}^{n+1}$ to the origin, a vector field $v$ along $f: M \to S^n$ gives rise to a map $\hat{v}: M \to \mathbf{R}^{n+1}$ with the property $\langle f, \hat{v} \rangle = 0$. The following lemma characterizes the elements of $K(f)$ in terms of the induced functions $\hat{v}$.

Lemma 4. *Let $v$ be a vector field along $f: M \to S^n$. Then $v \in K(f)$ if and only if $\Delta^M \hat{v} = 2e(f)\hat{v}$ holds, where $e(f) = \|f_*\|^2/2$ denotes the energy density of $f$.*

Proof. The covariant differentiation on $S^n$ can be obtained from that of $\mathbf{R}^{n+1}$ by performing the orthogonal projection to the corresponding tangent space of $S^n$ and thus, for $X \in \mathfrak{X}(M)$, we have

$$(\nabla_X v)^\wedge = X(\hat{v}) - \langle X(\hat{v}), f \rangle f,$$

where $X$ acts on $\hat{v}$ componentwise. An easy computation shows that

$$(\nabla_Y \nabla_X v)^\wedge = YX(\hat{v}) - \langle YX(\hat{v}), f \rangle f - \langle X(\hat{v}), f \rangle Y(f), \quad X, Y \in \mathfrak{X}(M),$$

i.e.

$$(\nabla^2 v)^\wedge = -\Delta^M \hat{v} + \langle \Delta^M \hat{v}, f \rangle f - \mathrm{trace}\, \langle d\hat{v}, f \rangle\, df$$

holds. On the other hand, we have

$$(\mathrm{trace}\, R(f_*, v) f_*)^\wedge = (\mathrm{trace}\, \langle f_*, v \rangle f_*)^\wedge - 2e(f)\hat{v} =$$

$$= \mathrm{trace}\, \langle df, \hat{v} \rangle df - 2e(f)\hat{v} = -\mathrm{trace}\, \langle f, d\hat{v} \rangle\, df - 2e(f)\hat{v}.$$

The identities yield that $v$ is a Jacobi vector field along $f$ if and only if

$$\Delta^M \hat{v} - \langle \Delta^M \hat{v}, f \rangle f = 2e(f)\hat{v} \tag{1}$$

is satisfied. Moreover, we have

$$\mathrm{trace}\, \langle f_*, \nabla v \rangle = \mathrm{trace}\, \langle df, d\hat{v} \rangle - \mathrm{trace}\, \langle d\hat{v}, f \rangle \langle df, f \rangle.$$

By $\|f\|^2=1$ the second term vanishes and so equation (ii) of Section 1 is equiva to the following

(2) $$\operatorname{trace}\langle df, d\theta\rangle = 0.$$

Further, harmonicity of $f$ means that $\Delta^M f=2e(f)f$ is valid and hence we

$$\langle \Delta^M \theta, f\rangle = -\langle \nabla^2 \theta, f\rangle = -\operatorname{trace}\nabla\langle d\theta, f\rangle + \operatorname{trace}\langle d\theta, df\rangle =$$

$$= \operatorname{trace}\nabla\langle \theta, df\rangle + \operatorname{trace}\langle d\theta, df\rangle = 2\operatorname{trace}\langle d\theta, df\rangle + \langle \theta, \Delta^M f\rangle = 2\operatorname{trace}\langle d\theta, dj$$

Assuming $v\in K(i)$ we obtain that $\langle \Delta^M \theta, f\rangle=0$ and hence (1) reduces to the e tion given in the lemma. Conversely, multiplying this equation with $f$ we $\langle \Delta^M \theta, f\rangle=0$ and hence (1) and (2) are satisfied which accomplishes the p

Corollary. *Let $f, f' : M\to S^n$ be orthogonal harmonic maps with $e(f)=e$ Then the (unique) vector field $v$ along $f$ with $\|v\|=1$ and* $\exp\circ((\pi/2)v)=j$ *a harmonic variation.*

Proof. By hypothesis $\theta=f_{\pi/2}=f'$ and harmonicity of $f'$ yields $\Delta$ $=2e(f')\theta=2e(f)\theta$. Applying the lemma above we obtain that $v\in K(f)$ whicl complishes the proof.

Remark. According to a result of [11] a vector field $v$ along $f$ is a harn variation if and only if $v$ is a Jacobi field along $f$ and $e(f_t)=e(f)$ holds fo $t\in \mathbf{R}$. Hence there is a one-to-one correspondence between the harmonic varia of $V_0(f)$ and the orthogonal pairs of harmonic maps $f, f' : M\to S^n$ with $e(f)=$

Now we turn to the variation space of the Veronese surface. Conside eigenspace $\mathscr{H}_2$ of the Laplacian $\Delta=\Delta^{S^2}$ of the Euclidean sphere $S^2$ correspo to the (second) eigenvalue $\lambda_2=6$ [1]. An element of $\mathscr{H}_2$ is the restriction (to of a homogeneous polynomial $p: \mathbf{R}^3\to\mathbf{R}$ of degree 2 which has the form

$$p = \sum_{k=1}^{3} a_k\varphi_k + 2\sum_{i<j} b_{ij}\varphi_{ij},$$

where $a_k, b_{ij}\in\mathbf{R}$ with $\sum_{k=1}^{3} a_k=0$ and $\varphi_k, \varphi_{ij}$, $k=1,2,3$, $1\le i<j\le 3$, are s on $S^2$ defined by $\varphi_k(x)=x_k^2$, $\varphi_{ij}(x)=x_ix_j$, $x=(x_1,x_2,x_3)\in S^2$. (cf. [1] p. in particular $\dim\mathscr{H}_2=5$.

Integration over $S^2$ defines a Euclidean scalar product on $\mathscr{H}_2$. Der $I=\|\varphi_k\|^2$ and $J=\|\varphi_{ij}\|^2$, the Veronese surface $f: S^2\to S^4$ is defined by

$$f(x_1,x_2,x_3) = \frac{N}{I-J}\sum_{k=1}^{3}\left(x_k^2-\frac{1}{3}\right)\varphi_k + \frac{2N}{J}\sum_{i<j} x_ix_j\varphi_{ij}, \quad (x_1,x_2,x_3)\in S^2,$$

where $N>0$ is a normalizing factor given by the condition $\|f\|=1$. Then full and homothetic [1]. It is well-known [1] that $f$ factors through the can projection $\pi: S^2\to\mathbf{R}P^2$ yielding an embedding of $\mathbf{R}P^2$ into $S^4$.

**Lemma 5.** *For the Veronese surface $f: S^2 \to S^4$, if $v \in K(f)$ then $\theta: S^2 \to \mathscr{H}_2$ is the decomposition*

$$\theta = \sum_{k=1}^{3} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},$$

*here* $a_k, b_{ij}$, $k=1, 3$, $1 \leqq i<j \leqq 3$, *are scalars on* $S^2$ *determined by the formulas*

$$a_1(x) = -\varepsilon x_2^2 + \varepsilon x_3^2 + 2\alpha_1 x_1 x_2 + 2\beta_1 x_1 x_3 - 2(\alpha_2+\beta_3) x_2 x_3,$$
$$a_2(x) = \varepsilon x_1^2 - \varepsilon x_3^2 + 2\beta_2 x_1 x_2 - 2(\beta_1+\alpha_3) x_1 x_3 + 2\alpha_2 x_2 x_3,$$
$$a_3(x) = -\varepsilon x_1^2 + \varepsilon x_2^2 - 2(\alpha_1+\beta_2) x_1 x_2 + 2\alpha_3 x_1 x_3 + 2\beta_3 x_2 x_3,$$
$$b_{12}(x) = -\frac{\alpha_1}{2} x_1^2 - \frac{\beta_2}{2} x_2^2 + \frac{\alpha_1+\beta_2}{2} x_3^2 - 2\gamma_1 x_1 x_3 + 2\gamma_2 x_2 x_3,$$
$$b_{23}(x) = \frac{\alpha_2+\beta_3}{2} x_1^2 - \frac{\alpha_2}{2} x_2^2 - \frac{\beta_3}{2} x_3^2 - 2\gamma_2 x_1 x_2 + 2\gamma_3 x_1 x_3,$$
$$b_{13}(x) = -\frac{\beta_1}{2} x_1^2 + \frac{\alpha_3+\beta_1}{2} x_2^2 - \frac{\alpha_3}{2} x_3^2 + 2\gamma_1 x_1 x_2 - 2\gamma_3 x_2 x_3,$$

$x=(x_1, x_2, x_3) \in S^2$, $\varepsilon, \alpha_k, \beta_k, \gamma_k \in \mathbf{R}$, $k=1, 2, 3$. *In particular*, $\dim K(f)=10$.

Proof. As $\theta$ maps into $\mathscr{H}_2$ we have the decomposition of $\theta$ as above with $\sum_{k=1}^{3} a_k=0$. On the other hand, Lemma 4 implies that

$$0 = \Delta\theta - 6\theta = \sum_{k=1}^{3} (\Delta a_k - 6a_k)\varphi_k + 2\sum_{i<j} (\Delta b_{ij} - 6b_{ij})\varphi_{ij}$$

and hence orthogonality of the polynomials $\varphi_{ij}$, $i<j$, and the relations $\langle\varphi_k, \varphi_{ij}\rangle=0$, $\langle\varphi_k, \varphi_r\rangle = J+\delta_{kr}(I-J)$, $k, r=1, 2, 3$, $i<j$, yield that the scalars $a_k, b_{ij}$, $k=1, 2, 3$, $i<j$, belong to $\mathscr{H}_2$. Thus

$$a_r = \sum_{k=1}^{3} a_k^r \varphi_k + 2\sum_{i<j} b_{ij}^r \varphi_{ij}, \quad r = 1, 2, 3,$$

and

$$b_{pq} = \sum_{k=1}^{3} a_k^{pq} \varphi_k + 2\sum_{i<j} b_{ij}^{pq} \varphi_{ij}, \quad 1 \leqq p < q \leqq 3,$$

where $a_k^r, b_{ij}^r, a_k^{pq}, b_{ij}^{pq} \in \mathbf{R}$ such that

$$(\mathrm{C}_1) \quad \sum_{k=1}^{3} a_k^r = 0 \quad \text{and} \quad \sum_{k=1}^{3} a_k^{pq} = 0, \quad r = 1, 2, 3, \ 1 \leqq p < q \leqq 3,$$

hold. Moreover, from the equation $\sum_{k=1}^{3} a_k = 0$ we obtain

$$(\mathrm{C}_2) \quad \sum_{r=1}^{3} a_k^r = 0 \quad \text{and} \quad \sum_{r=1}^{3} b_{ij}^r = 0.$$

Finally, the orthogonality relations for $\varphi_k$ and $\varphi_{ij}$ above imply that the con
$\langle f, \theta\rangle=0$ is equivalent to the equation

$$\sum_{k=1}^{3} a_k x_k^2 + 4 \sum_{i<j} b_{ij} x_i x_j = 0, \quad (x_1, x_2, x_3) \in S^2.$$

Substituting the explicit expressions of $a_k$ and $b_{ij}$ we get

$$\sum_{k=1}^{3} \sum_{r=1}^{3} a_k^r \varphi_k \varphi_r + 2 \sum_{i<j} \sum_{r=1}^{3} (b_{ij}^r + 2a_r^{ij}) \varphi_r \varphi_{ij} + 8 \sum_{i<j} \sum_{p<q} b_{ij}^{pq} \varphi_{ij} \varphi_{pq} = 0.$$

A straightforward computation, determining the coefficients of the fourth
homogeneous polynomial on the left hand side, shows that this equation is sat
if and only if the following relations hold:

(C$_3$) $a_k^k=0$ for $k=1, 2, 3$,

(C$_4$) $b_{12}^1+2a_1^{12}=b_{12}^2+2a_2^{12}=b_{13}^1+2a_1^{13}=b_{13}^3+2a_3^{13}=b_{23}^2+2a_2^{23}=b_{23}^3+2a_3^{23}=0$,

(C$_5$) $a_i^j+a_j^i+8b_{ij}^{ij}=0$ for $1 \leqq i<j \leqq 3$,

(C$_6$) $b_{23}^1+2a_1^{23}+4b_{12}^{13}+4b_{13}^{12}=b_{13}^2+2a_2^{13}+4b_{12}^{23}+4b_{23}^{12}=b_{12}^3+2a_3^{12}+4b_{13}^{23}+4b_{23}^{13}=$

Putting $\varepsilon=a_1^2$, the relations (C$_1$)—(C$_2$)—(C$_3$) imply that the matrix $A=(a_k^r)\in M($
has the form

$$A = \begin{bmatrix} 0 & \varepsilon & -\varepsilon \\ -\varepsilon & 0 & \varepsilon \\ \varepsilon & -\varepsilon & 0 \end{bmatrix}$$

and consequently, by (C$_5$), $b_{ij}^{ij}=0$ for $i<j$. Introducing the new (indepenc
variables

$$\begin{aligned} \alpha_1 &= b_{12}^1, & \alpha_2 &= b_{23}^2, & \alpha_3 &= b_{13}^3, \\ \beta_1 &= b_{13}^1, & \beta_2 &= b_{12}^2, & \beta_3 &= b_{23}^3, \\ \gamma_1 &= b_{12}^{13}, & \gamma_2 &= b_{23}^{12}, & \gamma_3 &= b_{13}^{23}, \end{aligned}$$

we see that all the remaining coefficients are expressible in terms of the varia
$\{\varepsilon, \alpha_k, \beta_k, \gamma_k \mid k=1, 2, 3\}$ and a straightforward computation leads to the coeffici
given in Lemma 5.

Our last result asserts that the Veronese surface is rigid. More precisely, we h
the following

**Theorem 2.** *For the Veronese surface $f: S^2 \to S^4$ the variation space V*
*is zero.*

*Proof.* Using the notations of Lemma 5 we parametrize $K(f)$ with
variables $\{\varepsilon, \alpha_k, \beta_k, \gamma_k \mid k=1, 2, 3\}$. Putting $v\in K(f)$ we have

$$\theta = \sum_{k=1}^{3} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},$$

where the coefficients $a_k, b_{ij}$, $k=1, 2, 3$, $1 \leqq i<j \leqq 3$, are given in Lemma 5.

Note that the parametrization of $K(f)$ is chosen in such a way as the cyclic permutation $\pi=(123)$ of the indices on the right hand sides will permute the scalars $a_1, a_2, a_3$ and $b_{12}, b_{23}, b_{13}$ cyclically. Now suppose, on the contrary, that $V(f)\neq\{0\}$, i.e. we may choose $v\in V(f)$ with $\|v\|^2=4J$. Then we have

$$4J = \|v\|^2 = \sum_{k=1}^{3}\sum_{r=1}^{3} a_k a_r \langle\varphi_k, \varphi_r\rangle + 4J \sum_{i<j} b_{ij}^2 = (I-J)\sum_{k=1}^{3} a_k^2 + 4J\sum_{i<j} b_{ij}^2,$$

or equivalently

$$1 = \frac{1}{2}\sum_{k=1}^{3} a_k^2 + \sum_{i<j} b_{ij}^2 \tag{3}$$

on $S^2$, where we used the equality $\dfrac{I-J}{4J}=\dfrac{1}{2}$ which can be obtained by integrating the polynomials $\varphi_3^2$ and $\varphi_{23}^2$ on $S^2$. Thus

$$(x_1^2+x_2^2+x_3^2)^2 = \frac{1}{2}\sum_{k=1}^{3} a_k(x_1, x_2, x_3)^2 + \sum_{i<j} b_{ij}(x_1, x_2, x_3)^2$$

is satisfied for all $(x_1, x_2, x_3)\in\mathbf{R}^3$. By computing the coefficients of the fourth order homogeneous polynomial on the right hand side we obtain a system of 15 quadratic equations in which the first 5 are given as follows

(i) $4\varepsilon^2+\alpha_1^2+\beta_1^2+(\alpha_2+\beta_3)^2=4$,

(ii) $\varepsilon(\alpha_1+2\beta_3)-\beta_1\gamma_1-(\alpha_2+\beta_3)\gamma_2=0$,

(iii) $-\varepsilon(\beta_1+2\alpha_3)+\alpha_1\gamma_1+(\alpha_2+\beta_3)\gamma_3=0$,

(iv) $\varepsilon(\alpha_2-\beta_3)+2(\alpha_1\beta_1-\beta_2(\beta_1+\alpha_3)-\alpha_3(\alpha_1+\beta_2))-\alpha_1\gamma_2+\beta_1\gamma_3-4\gamma_2\gamma_3=0$,

(v) $-2\varepsilon^2+4(\alpha_1^2+\beta_2^2+(\alpha_1+\beta_2)^2)+\alpha_1\beta_2-\beta_1(\beta_1+\alpha_3)-\alpha_2(\alpha_2+\beta_3)+8(\gamma_1^2+\gamma_3^2)=4$,

and, the equation (3) being invariant under the cyclic permutation $\pi=(123)$ of the indices, the last 10 equations are obtained from (i)—(v) by performing the index permutations $\pi$ and $\pi^2$. Denote the equations of the permuted systems by (i)$_\pi$—(v)$_\pi$ and (i)$_{\pi^2}$—(v)$_{\pi^2}$, respectively. Our purpose is to show that these equations have no solution. To do this, first denote by $s$ the symmetric polynomial given by $s(x, y)=x^2+xy+y^2$, $x, y\in\mathbf{R}$. Then (v) can be written as

$$-2\varepsilon^2+8s(\alpha_1, \beta_2)+(\alpha_1\beta_2-\beta_1^2-\beta_1\alpha_3-\alpha_2^2-\alpha_2\beta_3)+8(\gamma_1^2+\gamma_3^2) = 4.$$

Performing the index permutations $\pi$ and $\pi^2$ and adding these three equations we get

$$-6\varepsilon^2+7(s(\alpha_1, \beta_2)+s(\alpha_2, \beta_3)+s(\alpha_3, \beta_1))+16(\gamma_1^2+\gamma_2^2+\gamma_3^2) = 12.$$

In a similar way, from (i)—(i)$_\pi$—(i)$_{\pi^2}$ it follows that

$$12\varepsilon^2+2(s(\alpha_1, \beta_2)+s(\alpha_2, \beta_3)+s(\alpha_3, \beta_1)) = 12,$$

i.e. eliminating the terms containing the polynomial $s$ we have

(4)
$$24(1-\varepsilon^2)+8(\gamma_1^2+\gamma_2^2+\gamma_3^2) = 9.$$

On the other hand, fixing $\gamma_i$, $i=1,2,3$, the equations (ii)—(ii)$_\pi$—(ii)$_{\pi^2}$ and (i (iii)$_\pi$—(iii)$_{\pi^2}$ form a linear system for the variables $\alpha_i, \beta_i$, $i=1,2,3$. Denotin $M(\gamma_1,\gamma_2,\gamma_3)\in M(6,6)$ its matrix, we compute $\det M(\gamma_1,\gamma_2,\gamma_3)$. For $\xi, \eta,$ define

$$S(\xi,\eta,\zeta)=\begin{bmatrix} \varepsilon & 2\varepsilon & 0 & -\xi & -\eta & -\eta \\ -2\varepsilon & -\varepsilon & \xi & \xi & \eta & 0 \\ -\xi & -\xi & \varepsilon & 2\varepsilon & 0 & -\zeta \\ \xi & 0 & -2\varepsilon & -\varepsilon & \zeta & \zeta \\ 0 & -\eta & -\zeta & -\zeta & \varepsilon & 2\varepsilon \\ \eta & \eta & \zeta & 0 & -2\varepsilon & -\varepsilon \end{bmatrix}.$$

Permuting the rows and the coloumns of $M(\gamma_1,\gamma_2,\gamma_3)$ by the permutation we obtain $S(\gamma_1,\gamma_2,\gamma_3)$ and consequently $\det M(\gamma_1,\gamma_2,\gamma_3)=\det S(\gamma_1,\gamma_2,$ Similarly, by performing (135462) and (132465) on the rows and coloumn: $M(\gamma_1,\gamma_2,\gamma_3)$ we get $S(\gamma_2,\gamma_3,\gamma_1)$ and $S(\gamma_3,\gamma_1,\gamma_2)$ i.e. $\det S(\gamma_1,\gamma_2,\gamma$ $=\det S(\gamma_2,\gamma_3,\gamma_1)=\det S(\gamma_3,\gamma_1,\gamma_2)$. Thus, it is enough to compute $\det S(\xi,\eta$ To do this, let $S(\xi,\eta,\xi)$ have the decomposition

$$S(\xi,\eta,\zeta)=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A\in M(4,4)$. The matrix $A$ is centroskew and so by using a result o a direct computation shows that $\det A=(3\varepsilon^2-\xi^2)^2$. Assuming $3\varepsilon^2\neq\xi^2$ we have

$$\det S(\xi,\eta,\zeta) = \det A \det(D-CA^{-1}B) = 3\varepsilon^2(3\varepsilon^2-(\xi^2+\eta^2+\zeta^2))^2.$$

Suppose now that $\gamma_1^2=\gamma_2^2=\gamma_3^2=3\varepsilon^2$. Then equation (4) implies that $15+8\varepsilon^2$ which is impossible. Hence there exists $i\in\{1,2,3\}$ such that $\gamma_i\neq 3\varepsilon^2$. Then, the above, $\det M(\gamma_1,\gamma_2,\gamma_3)=3\varepsilon^2(3\varepsilon^2-(\gamma_1^2+\gamma_2^2+\gamma_3^2))^2$. Further, $\det M(\gamma_1,\gamma_2,\gamma_3)$ since otherwise $\gamma_1^2+\gamma_2^2+\gamma_3^2=3\varepsilon^2$ which contradicts to (4). Thus the linear sys in question has only trivial solution $\alpha_1=\alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=0$. Then equati (iv)—(iv)$_\pi$—(iv)$_{\pi^2}$ imply that two of the numbers $\gamma_1,\gamma_2,\gamma_3$ vanish. By equati (v)—(v)$_\pi$—(v)$_{\pi^2}$ we obtain $\varepsilon=0$ which again contradicts to (4).

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