ON RIGIDITY OF HARMONIC MAPPINGS INTO SPHERES

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1. *Introduction*

A map $f: M \to M'$ of a compact oriented Riemannian manifold M into a complete Riemannian manifold M' is harmonic if its energy $E(f) = \frac{1}{2} \int_M \|f_*\|^2 \operatorname{vol}(M)$ is stable to first order with respect to variations of f [**2**]. Though, by the work of T. Sunada [**14**], harmonic maps into nonpositively curved codomains are globally rigid, an essential obstruction to proving rigidity in the case when $\mathrm{Riem}^{M'} \geqslant 0$ is the lack of Hartman's uniqueness [**3**]. On the other hand it has long been noticed that, in all cases, harmonic maps behave nicely with respect to infinitesimal deformations preserving harmonicity up to second order, that is, the second variation formula [**8**, **13**] and Jacobi fields along harmonic maps have been proved to be useful in showing rigidity of harmonic maps (see [**2**, **6**, **9**, **10**, **11**, **12**, **13**]).

The purpose of this paper is to study harmonic maps into spheres with various rigidity properties when higher order terms of the expansion of the energy functional along a variation are also taken into account. In Section 2 we define the (geodesic) variations of a harmonic map given by translating the map along geodesics of a prescribed Jacobi field along this map. The concept of harmonic variation is introduced [**15**] and its close relationship with the Jacobi fields is indicated (Theorem 1). In Section 3 infinitesimal and local rigidity of harmonic maps are studied in detail; for example, by reducing the problem to that of the linear algebra, we show that harmonic embeddings $f: S^m \to S^n$ with energy density $e(f) = m/2$ are rigid (Theorem 2). (For examples of nonrigid harmonic embeddings, see [**17**].) In Section 4 certain metric spaces of locally rigid harmonic maps are introduced and their classification is reduced to that of the canonical inclusion map $i: S^m \to S^n$.

Throughout this note all manifolds, maps, bundles, etc. will be smooth, that is, of class C^∞. The report [**2**] is our general reference, adopting the sign conventions of [**5**].

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2. *Jacobi fields and harmonic variations*

Let $f: M \to S^n$ be a map of a compact oriented Riemannian manifold M of dimension m into the Euclidean n-sphere S^n. A vector field v along f, that is, a section of the pull-back bundle $F = f^*(T(S^n))$, gives rise to a (geodesic, 1-parameter)

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variation $t \to f_t = \exp \circ (tv)$, $t \in \mathbb{R}$. Then $v \in C^\infty(F)$ is said to be a harmonic variation if the maps $f_t : M \to S^n$ are harmonic [2] for all $t \in \mathbb{R}$. The set of all harmonic variations (or the variation space) of the given map f is denoted by $V(f) \subset C^\infty(F)$.

If $t \to f_t$, $t \in \mathbb{R}$, is an arbitrary (not necessarily geodesic) variation of f through harmonic maps then $v = \left.\frac{\partial f_t}{\partial t}\right|_{t=0}$ is a Jacobi field along f [2], that is, v is a solution of the Jacobi equation

$$J_f v = -\nabla^2 v + \operatorname{trace} R(f_*, v) f_* = 0,$$

where ∇ is the induced connection of the bundle $\xi = F \otimes \Lambda^*(T^*(M))$; $\nabla^2 v$ is the trace of the bilinear form $(X, Y) \to \nabla_X \nabla_Y v - \nabla_{\nabla_X Y} v$ (see [11]); $f_* \in C^\infty(\xi)$ is the differential of f and R denotes the Riemannian curvature tensor of S^n. The differential operator $J_f \in \operatorname{Diff}_2(F, F)$ is strongly elliptic [11] over the compact manifold M, and hence solutions of the Jacobi equation have uniqueness in the Cauchy problem; the nullity $\operatorname{Null}(f) = \dim J(f)$, $J(f) = \ker J_f$, and the Morse index of f (that is, card $\{\text{eigenvalues}(J_f) < 0\}$) are finite. By a recent theorem of P. F. Leung [6] the index of a nonconstant harmonic map $f : M \to S^n$, with $n \geqslant 3$, is strictly positive; by an earlier result of R. T. Smith, $\operatorname{index}(id_{S^n}) = n+1$ [13]. For the nullity of id_{S^n} see [13] and Section 3 below.

THEOREM 1. *Let v be a Jacobi field along the harmonic map $f : M \to S^n$ and write*

$$T(v) = \{t \in \mathbb{R} \mid f_t = \exp \circ (tv) \text{ is harmonic}\}.$$

Then the following cases can occur:

(1) *$T(v)$ consists of at most two points (including 0) and $\|v\|$ is not constant,*

(2) *$\|v\|$ is a non-zero constant and $T(v) = \frac{\pi}{\|v\|}\mathbb{Z}$,*

(3) *$T(v) = \mathbb{R}$, that is, $v \in V(f)$, and this is the case if and only if $\|v\|$ is constant and* $\operatorname{trace}\langle f_*, \nabla v\rangle = 0$.

Proof. Given $0 \neq v \in C^\infty(F)$ the induced variation

$$t \longrightarrow (f_t)_* \in C^\infty(F^t \otimes T^*(M)), \qquad F^t = (f_t)^*(T(S^n)),$$

can be conveniently described by considering the 1-forms

$$P_v(t) = (\tau_0^t[v] \otimes id_{T^*(M)})(f_t)_* \in C^\infty(F \otimes T^*(M))$$

with values in F, where $\tau_{t'}^{t''}[v] : F^{t'} \to F^{t''}$ is the isomorphism defined by the parallel transport along the geodesic segments $t \to f_t(x)$, $t \in [t', t'']$ (or $t \in [t'', t']$), $x \in M$. For simplicity, we write $\tau_{t'}^{t''} = \tau_{t'}^{t''}[v] \otimes id_{\Lambda^*(T^*(M))}$ and omit 0 in $f_0, \tau_0^{t'}$, etc. Then $t \to (f_t)_* X_x$, with $X_x \in T_x(M)$, $x \in M$, is a Jacobi field along $t \to f_t(x)$, $t \in \mathbb{R}$, and applying τ^t to both sides of the corresponding Jacobi equation and omitting X_x we obtain

$$\text{(A)} \qquad \frac{d^2 P_v(t)}{dt^2} + R(P_v(t), v)v = 0$$

with initial data

$$\text{(B)} \qquad P_v(0) = f_* \quad \text{and} \quad \left.\frac{dP_v(t)}{dt}\right|_{t=0} = \nabla v$$

(cf. [**15**, **16**]). Using the well-known formula [**5**, Vol. I, p. 203], we have

$$R(X, Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y, \qquad X, Y, Z \in \mathfrak{X}(S^n);$$

the initial value problem (A) with (B) can be solved explicitly

$$P_v(t) = \left\langle f_* + t\nabla v, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|} + \cos(t\|v\|)\left(f_* - \left\langle f_*, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}\right)$$
$$+ \frac{\sin(t\|v\|)}{\|v\|}\left(\nabla v - \left\langle \nabla v, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}\right)$$

(in the case when $v_x = 0$ we take $P_v(t) = f_* + t\nabla v$ at $x \in M$). In order to compute the tension field $\tau(f_t) = \operatorname{div}(f_t)_* \in C^\infty(F^t)$ [**2**] in terms of f_* and v we need the identity

$$(\tau^t \circ \nabla^t \circ \tau_t)w - \nabla w = -R\left(\int_0^t P_v(s)ds, v\right)w, \qquad w \in C^\infty(F),$$

relating the induced connections ∇^t and ∇ on $\xi^t = F^t \otimes \Lambda^*(T^*(M))$ and ξ, respectively. (This can be shown directly by using local symmetricity of S^n and the definition of R by parallel transport; see also [**15**].) Then, as in [**16**], we get

$$\tau^t(\tau(f_t)) = \tau^t \operatorname{trace}\{\nabla^t(f_t)_*\} = \operatorname{trace}\{(\tau^t \circ \nabla^t \circ \tau_t)P_v(t)\}$$
$$= -\left(\nabla^* P_v(t) + \operatorname{trace}\left\{R\left(\int_0^t P_v(s)ds, v\right)P_v(t)\right\}\right);$$

in particular, $f_t : M \to S^n$ is harmonic if and only if

$$\text{(C)} \qquad \Psi(v, t) = \nabla^* P_v(t) + \operatorname{trace}\left\{R\left(\int_0^t P_v(s)ds, v\right)P_v(t)\right\} = 0.$$

On substituting the solution $P_v(t)$ *of* (A), (B) into (C) an admittedly long calculation yields

$$\Psi(v,t) = \cos(t\|v\|)\nabla^* f_* - \frac{\sin(t\|v\|)}{\|v\|}\nabla^2 v$$

$$+ \frac{\sin(t\|v\|)}{\|v\|}\operatorname{trace}\{R(f_*, v)f_*\}$$

$$+ \frac{2t\sin(t\|v\|)}{\|v\|}\operatorname{trace}\{R(f_*, v)\nabla v\}$$

$$+ \frac{2\sin(t\|v\|) - 2t\|v\|\cos(t\|v\|)}{\|v\|^3}\operatorname{trace}\{R(\nabla v, v)\nabla v\}$$

$$- \frac{\cos(t\|v\|) - 1}{\|v\|^2}\langle\nabla^* f_*, v\rangle v + \frac{\sin(t\|v\|) - t\|v\|}{\|v\|^3}\langle\nabla^2 v, v\rangle v$$

$$+ \frac{\sin(2t\|v\|) - 2\sin(t\|v\|)}{2\|v\|^3}\operatorname{trace}\{\langle R(f_*, v)f_*, v\rangle\} v$$

$$+ \frac{-\cos(2t\|v\|) - 2t\|v\|\sin(t\|v\|) + 1}{\|v\|^4}\operatorname{trace}\{\langle R(f_*, v)\nabla v, v\rangle\} v$$

$$+ \frac{-\sin(2t\|v\|) - 4\sin(t\|v\|) + 4t\|v\|\cos(t\|v\|) + 2t\|v\|}{2\|v\|^5}\operatorname{trace}\{\langle R(\nabla v, v)\nabla v, v\rangle\} v$$

(in the case when $v_x = 0$ we take the corresponding limits). Assume now that $f: M \to S^n$ is harmonic and $v \in J(f)$. By Cauchy uniqueness, the complement M' of the zero locus of $\|v\|$ (not identically zero) is dense (and open) in M. Let $0 \neq t \in T(v)$ be fixed. Using the Jacobi equation and the explicit formula for $\Psi(v,t)$ above, a routine calculation shows that the equation

$$\langle\Psi(v,t), v\rangle = \frac{1-\cos(2t\|v\|)}{\|v\|^2}\operatorname{trace}\langle R(f_*, v)\nabla v, v\rangle$$

$$+ \frac{\sin(2t\|v\|) - 2t\|v\|}{2\|v\|^3}\{\|v\|^2\langle\nabla^2 v, v\rangle - \operatorname{trace}\langle R(\nabla v, v)\nabla v, v\rangle\} = 0$$

holds on M'. If $0 \neq t' \in T(v)$ we get

$$\left(\frac{1-\cos(2t\|v\|)}{\sin(2t\|v\|) - 2t\|v\|} - \frac{1-\cos(2t'\|v\|)}{\sin(2t'\|v\|) - 2t'\|v\|}\right) 2\|v\| \operatorname{trace}\langle R(f_*, v)\nabla v, v\rangle = 0 \quad (\text{on } M').$$

Write $U \subset M'$ for the (possibly empty) complement of the zero locus of $\operatorname{trace}\langle R(f_*, v)\nabla v, v\rangle$. Then

$$\frac{1-\cos(2t\|v\|)}{\sin(2t\|v\|) - 2t\|v\|} = \frac{1-\cos(2t'\|v\|)}{\sin(2t'\|v\|) - 2t'\|v\|}$$

is valid on U. Assuming that $\|v|U\|$ is non-constant, we can choose a non-empty open interval $I \subset \operatorname{im}(\|2v|U\|) \subset \mathbb{R}_+$ such that

$$\frac{1-\cos(t\lambda)}{\sin(t\lambda)-t\lambda} = \frac{1-\cos(t'\lambda)}{\sin(t'\lambda)-t'\lambda}$$

for all $\lambda \in I$. Both sides of this equation are analytic in λ and thus the equality holds for all $\lambda \neq 0$. Comparison of the least positive roots gives $t = t'$, that is, $T(v)$ consists of at most two points $\{0, t\}$ and we have case (1) of the theorem. If, on the other hand, $\|v|U\|$ is constant, then, in particular, $\nabla^2(\|v\|^2) \geqslant 0$ on U. Now the formula for $\langle \Psi(v, t), v\rangle$ above shows that $x \in M' - U$ implies that

$$\begin{aligned} 0 &= \|v_x\|^2 \langle \nabla^2 v, v\rangle_x - \operatorname{trace}\langle R(\nabla v, v)\nabla v, v\rangle_x \\ &= \tfrac{1}{2}\|v_x\|^2 (\nabla^2(\|v\|^2))_x - \operatorname{trace}\langle \nabla v, v\rangle_x^2 , \end{aligned}$$

using the formula for R above, and so $\nabla^2(\|v\|^2) \geqslant 0$ on $M' - U$. Since $M' \subset M$ is dense we get $\nabla^2(\|v\|^2) \geqslant 0$ on M and, by Hopf's lemma [**5**, Vol. II, p. 338], compactness of M implies that $\|v\|$ is constant. Without loss of generality we may assume that $\|v\| = 1$. Since the antipodal map of S^n is an isometry, $\pi Z \subset T(v)$ is always satisfied. Thus either $T(v) = \pi Z$ and we have case (2), or there exists $t_0 \in (0, \pi) \cap T(v)$. Then

$$\langle \Psi(v, t_0), v\rangle = \big(\cos(2t_0) - 1\big)\operatorname{trace}\langle f_*, \nabla v\rangle = 0$$

and so $\operatorname{trace}\langle f_*, \nabla v\rangle = 0$. Using this and the fact that $\|v\|$ is constant, one can easily check that $\Psi(v, t) = 0$ for all $t \in \mathbb{R}$, that is, $v \in V(f)$ and we have case (3). The proof is complete.

REMARK 1. If $v \in J(f)$ and $f_t = \exp \circ (tv)$, $t \in \mathbb{R}$, is harmonic in the direction $\tau_t v$, that is, if $\langle \tau(f_t), \tau_t v\rangle = 0$, then $v \in V(f)$. Indeed, the hypothesis implies that $\langle \Psi(v, t), v\rangle = 0$ and we can argue as above.

REMARK 2. The Euler class of the tangent bundle of an even sphere is twice the generator which, in the case when $m = n$ and is even, yields that a harmonic map $f : M \to S^n$ with nontrivial variation space $V(f)$ must have Brouwer degree zero.

REMARK 3. The energy $E(f_t)$ is constant along a harmonic variation $t \to f_t = \exp \circ (tv)$, $v \in V(f)$, since f_t is a critical point of E for all $t \in \mathbb{R}$. In fact, a more precise statement appears in [**16**], namely, that $v \in V(f)$ if and only if $v \in J(f)$ and the energy density $e(f_t) = \frac{1}{2}\operatorname{trace}\|(f_t)_*\|^2$, as a scalar on M, is the same for all $t \in \mathbb{R}$.

3. *Rigidity*

Let $f : M \to S^n$ be a harmonic map and put

$$K(f) = \{v \in J(f) \mid \operatorname{trace}\langle f_*, \nabla v\rangle = 0\}.$$

Then $K(f) \subset J(f)$ is a linear subspace and, by Theorem 1,

$$V(f) = \{v \in K(f) \mid \|v\| \text{ is constant}\}.$$

In what follows we compute $K(i)$, where $i : S^m \to S^n$, $m \leqslant n$, is the inclusion map. By $S^n \subset \mathbb{R}^{n+1}$, vectors tangent to S^n are identified with their translates at the origin. Then the canonical base vectors $e_{m+2}, \ldots, e_{n+1} \in \mathbb{R}^{n+1}$ define k $(= n-m)$ orthonormal parallel sections $W^1, \ldots, W^k$ of the normal bundle of i. Assuming that $v \in K(i)$, the (orthogonal) decomposition

$$v = W + \sum_{j=1}^{k} \langle v, W^j \rangle W^j, \qquad W \in \mathfrak{X}(S^m),$$

splits the Jacobi equation $J_f v = 0$ into the system

$$\Delta \langle v, W^j \rangle + m \langle v, W^j \rangle = 0, \qquad j = 1, \ldots k,$$

$$\nabla^2 W + (m-1) W = 0$$

(cf. [**16**]). It follows that since the scalar $\langle v, W^j \rangle$ is an eigenfunction of the Laplacian on S^m, it is the restriction of a homogeneous linear function on $\mathbb{R}^{m+1}$ [**1**], that is, there exists a unique vector $b_j \in \mathbb{R}^{m+1}$ with $\langle v_x, W^j_x \rangle = \langle b_j, x \rangle$, $x \in S^m$. Moreover, denoting by β the 1-form on S^m that corresponds to W by duality, the last equation can be rewritten as

$$\Delta \beta - 2(m-1)\beta = 0.$$

On the other hand, $0 = \operatorname{trace} \langle i_*, \nabla v \rangle = \operatorname{trace} \langle i_*, \nabla W \rangle = -\nabla^* \beta$ and a result of A. Lichnerowicz [7, Proposition 1, p. 80], shows that $W \in so(m+1)$ is a Killing vector field. The converse is obvious; thus we obtained the following.

LEMMA 1. *For the canonical embedding $i : S^m \to S^n$ the vector field v along i belongs to $K(i)$ if and only if the tangential part W of v is a Killing vector field and there exist vectors $b_1, \ldots, b_k \in \mathbb{R}^{m+1}$, $k = n-m$, such that*

$$v_x = W_x + \sum_{j=1}^{k} \langle b_j, x \rangle W^j_x, \qquad x \in S^m,$$

where $\{W^1, \ldots, W^k\}$ is the canonical orthonormal system of parallel sections of the normal bundle of i.

In particular, $\dim K(i) = \dfrac{m(m+1)}{2} + (n-m)(m+1)$ and $K(id_{S^n}) = so(n+1)$. The last equality, in the case when $n \neq 2$, follows from a sharper result of R. T. Smith [**13**], in which he showed that $\operatorname{Null}(id_{S^n}) = n(n+1)/2$ or, equivalently, that $J(id_{S^n}) = so(n+1)$, if $n \neq 2$. Note, however, that, by the presence of conformal diffeomorphisms, $\dim J(id_{S^2}) = 6$ (cf. [**12**]). Furthermore, using Lemma 1, we have the following.

PROPOSITION. *If $f : M \to S^n$ is a harmonic Riemannian submersion and $v \in K(f)$ is* *projectable, that is, if* $f(x) = f(x')$ *implies that* $v_x = v_{x'}$, $x, x' \in M$, *then* $v = X \circ f$, *where* $X \in so(n+1)$. *If, moreover,* $v \in V(f)$ *then* $f_t = \exp \circ (tv) = \phi_t \circ f$, $t \in \mathbb{R}$, *where* (ϕ_t) *is the* 1*-parameter group of isometries induced by* X.

Proof. As v takes its values in $T(S^n)$ there is a unique vector field X on the codomain S^n such that $v = X \circ f$ is valid. By choosing suitable local orthonormal frames in M and in S^n we obtain $\nabla^2 v = \nabla^2(X \circ f) = (\nabla^2 X) \circ f$ (cf. [17, Proposition 2]) and hence

$$(\nabla^2 X) \circ f = \operatorname{trace} R(f_*, v) f_* = \left(\operatorname{trace} R\big((id_{S^n})_*, X\big)(id_{S^n})_*\right) \circ f.$$

Similarly, $\operatorname{trace}\langle f_*, \nabla v\rangle = \operatorname{trace}\langle (id_S n)_*, \nabla X\rangle \circ f$. These two equations imply that $X \in K(id_{S^n})$ and so, applying Lemma 1 in the case when $m = n$, we get $X \in so(n+1)$. If $v \in V(f)$ then $\|X\|$ is constant and thus for any vector field Y on S^n we have

$$\langle Y, \nabla_X X\rangle = -\langle X, \nabla_Y X\rangle = -\tfrac{1}{2} Y\|X\|^2 = 0$$

(cf. [5]). It follows that $\nabla_X X = 0$ on S^n or equivalently that every integral curve $t \to \phi_t(x)$, $t \in \mathbb{R}$, $x \in S^n$, of X is a geodesic. Hence

$$f_t(x) = \exp(tv_x) = \exp(tX_{f(x)}) = \phi_t(f(x))$$

for all $x \in M$; this completes the proof.

A harmonic map $f : M \to S^n$ is said to be *infinitesimally rigid* if for every projectable $v \in K(f)$ there exists $X \in so(n+1)$ such that the equation $v = X \circ f$ holds. (Note that $X \circ f \in K(f)$ is always satisfied for $X \in so(n+1)$). The map f is *locally rigid* if for every projectable harmonic variation v there exists a 1-parameter subgroup $(\phi_t) \subset O(n+1)$ such that $f_t = \exp \circ (tv) = \phi_t \circ f$ for all $t \in \mathbb{R}$. (Obviously, isometries of the codomain preserve infinitesimal and local rigidity.) The reason for the use of $K(f)$ to denote infinitesimal rigidity is that the set of all infinitesimal isometries of S^n precomposed by f forms a linear space which suggests a choice of $K(f)$ defined by linear constraints. As for local rigidity, for projectable $v = X \circ f \in K(f)$, we cannot expect $\exp \circ (tX \circ f) = \phi_t \circ f$ to be valid for all $t \in \mathbb{R}$, where (ϕ_t) denotes the 1-parameter group of isometries induced by X. Hence, in the definition of local rigidity, we have to restrict ourselves to the subset $V(f) \subset K(f)$ (which is not a linear subspace in general, cf. [17]). Note that in all the known examples, infinitesimal and local rigidities are equivalent. By the above proposition, any harmonic Riemannian submersion is infinitesimally and locally rigid.

THEOREM 2. *The canonical inclusion map* $i : S^m \to S^n$ *is infinitesimally and locally rigid.*

Proof. Let $v \in K(i)$ be fixed and consider the decomposition of v in Lemma 1. For $p, q \in \mathbb{N}$, let $M(p, q)$ be the vector space of $(p \times q)$-matrices; $M(p, q) = L(\mathbb{R}^q, \mathbb{R}^p)$. Finally, write $B \in M(k, m+1)$, $k = n-m$, for the matrix whose rows are $b_1, \dots, b_k \in \mathbb{R}^{m+1}$, and which occurs in the decomposition of v. By identifying the tangent vectors of $S^n \in \mathbb{R}^{n+1}$ with their translates at the origin, the vector field v along f gives rise to a matrix $A \in M(n+1, m+1)$ with upper block $W \in so(m+1)$

and lower block $B \in M(k, m+1)$ such that $v_x = Ax$ holds for all $x \in S^m$. If $Y \in so(k)$ is *any* skew-symmetric matrix then

$$X = \left[\begin{array}{c|c} W & -B^T \\ \hline B & Y \end{array}\right] \in so(n+1) \tag{D}$$

is a Killing vector field on S^n with $v = X \circ i$ and hence $i : S^m \to S^n$ is infinitesimally rigid.

Now let $v \in V(i)$ with $\|v\| = 1$. Then, for $x \in S^m$, we have

$$1 = \|v\|^2 = \|Ax\|^2 = \langle A^T Ax, x\rangle = \langle (W^T W + B^T B)x, x\rangle,$$

that is, we get $W^T W + B^T B = I_{m+1}$ (the identity). (In particular, by the identification $K(i) = so(m+1) \times M(k, m+1)$ the variation space of i has the form $V(i) = \mathbb{R}\{(W, B) \in K(i) \mid -W^2 + B^T B = I_{m+1}\}$.) To prove local rigidity, we have to find $Y \in so(k)$ such that the corresponding Killing vector field X above has the property that the integral curve $t \to \phi_t(x) = e^{tX}x$ of X is a geodesic for all $x \in S^m$. This holds if and only if $\frac{d^2}{dt^2} e^{tX}x = -e^{tX}x$, that is, if and only if $X^2 x = -x$ for all $x \in S^m$. By computing X^2 in terms of the blocks of X and using the fact that $-W^2 + B^T B = I_{m+1}$, we have to choose $Y \in so(k)$ so as to satisfy the equality

$$BW + YB = 0.$$

Let $e_1, \ldots, e_{m+1} \in \mathbb{R}^{m+1}$ be the canonical base vectors and let $c_1, \ldots, c_{m+1} \in \mathbb{R}^k$ denote the columns of B. Having in mind that the above equation must be valid we define $Yc_j = -BWe_j$, $j = 1, \ldots, m+1$. Then Y is skew on the vectors $c_1, \ldots, c_{m+1}$ since

$$\begin{aligned}
\langle Yc_j, c_{j'}\rangle &= -\langle BWe_j, Be_{j'}\rangle = -\langle We_j, B^T Be_{j'}\rangle = -\langle We_j, (W^2 + I_{m+1})e_{j'}\rangle \\
&= \langle e_j, W(W^2 + I_{m+1})e_{j'}\rangle = \langle e_j, (W^2 + I_{m+1})We_{j'}\rangle \\
&= \langle (W^2 + I_{m+1})e_j, We_{j'}\rangle = \langle B^T Be_j, We_{j'}\rangle = \langle Be_j, BWe_{j'}\rangle \\
&= -\langle c_j, Yc_{j'}\rangle, \qquad j, j' = 1, \ldots, m+1.
\end{aligned}$$

Putting $V = \operatorname{span}\{c_1, \ldots, c_{m+1}\} \subset \mathbb{R}^{m+1}$ we define

$$Yx = \sum_{j=1}^{m+1} \alpha_j Yc_j = -\sum_{j=1}^{m+1} \alpha_j B(We_j) \in V,$$

where $x = \sum_{j=1}^{m+1} \alpha_j c_j \in V$. We have

$$\begin{aligned}
\langle Yx, c_{j'}\rangle &= \left\langle \sum_{j=1}^{m+1} \alpha_j Yc_j, c_{j'} \right\rangle = \sum_{j=1}^{m+1} \alpha_j \langle Yc_j, c_{j'}\rangle \\
&= -\sum_{j=1}^{m+1} \alpha_j \langle c_j, Yc_{j'}\rangle = -\langle x, Yc_{j'}\rangle
\end{aligned}$$

and so Y is well defined (that is, Yx does not depend on the particular decomposition of x) and skew on V. Extending $Y|V$ to a skew-symmetric linear endomorphism $Y \in so(m+1)$ the equation $BW+YB = 0$ holds; this completes the proof.

REMARK 1. An immediate consequence of Theorem 2 is that any harmonic Riemannian submersion $f: M \to S^n$ onto a totally geodesic submanifold of S^n is infinitesimally and locally rigid.

Indeed, assuming that $\operatorname{im} f = S^m \subset S^n$, any projectable vector field v along f can be written uniquely as $v = w \circ f$, where w is a vector field along the inclusion $i: S^m \to S^n$. If, moreover, $v \in K(f)$ then by choosing suitable local orthonormal frames a similar computation as in the proof of Proposition 2 in [17] shows that

$$\nabla^2 v = (\nabla^2 w) \circ f = \operatorname{trace} R(f_*, v) f_* = (\operatorname{trace} R(i_*, w) i_*) \circ f$$

and trace $\langle f_*, \nabla v\rangle = (\operatorname{trace}\langle i_*, \nabla w\rangle) \circ f$, that is, $w \in K(i)$. Thus Theorem 2 implies the existence of a vector field $X \in so(n+1)$ with $X \circ i = w$ and it follows that $X \circ f = v$, which completes the proof in the infinitesimal case. As for local rigidity, if $v \in V(f)$ is as above then $w \in V(i)$ and, again by Theorem 2, we can choose $X \in so(n+1)$ so as to satisfy $\nabla_X X \circ i = 0$. Thus $\nabla_X X \circ f = 0$ which implies local rigidity of f.

REMARK 2. From the proof of Theorem 2 we see that $V(id_{S^n}) = so(n+1) \cap \mathbb{R}SO(n+1)$, in particular, $V(id_{S^n}) = \{0\}$ for n even. (This follows also from Remark 2 after Theorem 1.)

For odd spheres a stronger version of Theorem 2 is valid. First we state the following lemma, due to A. Lee.

LEMMA 2. *Let $W \in so(m+1)$ and $B \in M(k, m+1)$ with $-W^2 + B^T B = I_{m+1}$. If $n = k+m$ is odd then the equation $BW + YB = 0$ has a solution Y such that the corresponding matrix X in* (D) *belongs to $so(n+1) \cap O(n+1)$.*

Proof. We verify the statement for $k \geqslant m+1$ and $m = 2r-1$, $r \in \mathbb{N}$. (The other cases can be treated similarly.) By the singular values decomposition of matrices, there exist $V \in O(k)$ and $U \in O(m+1)$ such that $B = V\Sigma U^T$, where

$$\Sigma = \left[\begin{array}{ccc} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_{2r} \\ \hline & 0 & \end{array}\right] \in M(k, m+1).$$

and $\rho_1 \geqslant \ldots \geqslant \rho_{2r}$. (The numbers ρ_j, $j = 1, \ldots, 2r$, are often called the singular values of B.) Then $B^T B = U \operatorname{diag}(\rho_1^2, \ldots, \rho_{2r}^2) U^T$ and using the equation $-W^2 + B^T B = I_{m+1}$ we have $-W^2 = U \operatorname{diag}(1-\rho_1^2, \ldots, 1-\rho_{2r}^2) U^T$. On the other hand, U diagonalizes W^TW and hence, by an appropriate choice of U, we have $W = U\Lambda U^T$ where

$$\Lambda = \operatorname{diag}\left(\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \lambda_r \\ -\lambda_r & 0 \end{bmatrix}\right) \in so(m+1).$$

Thus, computing $W^2 = U\Lambda^2U^T$, we get $\rho_{2j-1} = \rho_{2j}$, that is, $\lambda_j^2 + \sigma_j^2 = 1$, where $\sigma_j = \rho_{2j}$, $j = 1, \dots, r$. Because $k-(m+1) = k-2r$ is even we can define $Y = V(\Lambda \oplus E)V^T \in so(k)$, where

$$E = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \in so(k-2r).$$

Then $BW + YB = V(\Sigma\Lambda - E\Sigma)U^T = 0$ and a staightforward computation shows that the matrix X in (D) is orthogonal; this completes the proof.

THEOREM 3. *If n is odd and v is a harmonic variation of the canonical inclusion map $i : S^m \to S^n$ then there exists a Killing vector field $X \in so(n+1)$ of constant norm on S^n such that $v = X \circ i$.*

Proof. Keeping the notation of the proof of Theorem 2, let $v \in V(i)$, where $\|v\|$ is a constant, say 1, and write $W \in so(m+1)$ and $B \in M(k, m+1)$ for the matrices defined by v. Choosing $Y \in so(k)$ as in Lemma 2 we have $v = X \circ i$ and $\|X_x\|^2 = \|Xx\|^2 = \|x\|^2 = 1$ for all $x \in S^n$. Thus the theorem is proved.

REMARK. Theorem 2 can also be proved by using the singular values decomposition; however, we preferred to give an elementary and direct construction for a solution of the equation $BW + YB = 0$ in $so(k)$.

Returning to the general situation, let $f : M \to S^n$ be an infinitesimally rigid harmonic map and assume that $\operatorname{im} f \subset S_0 = \exp\left(f_*(T_{x_0}(M)\right)$ for some $x_0 \in M$. If $v \in V(f)$ is projectable then infinitesimal rigidity of f implies the existence of a Killing vector field $X \in so(n+1)$ with $v = X \circ f$. Moreover, as $\|X \circ f\| = \|v\|$ is constant it follows that $\|X\|$ is constant on a relatively open neighbourhood of x_0 in S_0. Thus $\|X|S_0\|$ is constant, that is, $X \circ i_0 \in V(i_0)$, where $i_0 : S_0 \to S^n$ is the canonical inclusion. By local rigidity of i_0 there exists $\tilde{X} \in so(n+1)$ such that $\tilde{X} \circ i_0 = X \circ i_0$ and the integral curves $t \to \tilde{\phi}_t(x)$, $t \in \mathbb{R}$, of $\tilde{X}$ are geodesics for all $x \in S_0$. Hence f is locally rigid and we have proved the following.

THEOREM 4. *Let $f : M \to S^n$ be an infinitesimally rigid harmonic map with $\operatorname{im} f \subset \exp\left(f_*(T_{x_0}(M))\right)$ for some $x_0 \in M$. Then f is locally rigid.*

The condition for the image of f is satisfied if either f has maximal rank, with surjective f_*, at some $x_0 \in M$, or f is totally geodesic. (In the latter case f maps geodesics onto geodesics and a straightforward argument shows that $\operatorname{im} f \subset S^n$ is a totally geodesic submanifold.)

4. *Metric spaces of locally rigid harmonic embeddings*

Write $\operatorname{Harm}(M, S^n)$ for the set of all harmonic maps $f : M \to S^n$. Two maps f

and f' in $\mathrm{Harm}(M, S^n)$ are called equivalent, $f \sim f'$, if there exist maps $f^0, \ldots, f^{k+1} \in \mathrm{Harm}(M, S^n)$ and vectors $v^j \in V(f^j)$, $j = 0, \ldots, k$, such that $f^0 = f$, $f^{k+1} = f'$ and $\exp \circ v^j = f^{j+1}, j = 0, \ldots, k$. Clearly $\sim$ is an equivalence relation. If $f \sim f'$, write $\rho(f, f')$ for the infimum of the numbers $\sum_{j=0}^{k} \|v^j\|$ for which $v^j \in V(f^j)$, $\exp \circ v^j = f^{j+1}$, $j = 0, \ldots, k$ and $f^0 = f$, $f^{k+1} = f'$. Thus ρ is a distance function on the equivalence class $N(f) \subset \mathrm{Harm}(M, S^n)$ containing the harmonic map $f : M \to S^n$.

Our present aim is to determine to what extent $N(f)$ depends on the particular choice of f. In this way, for locally rigid harmonic embeddings $f : M \to S^n$ (where $V(f')$ is the same for all $f' \in N(f)$), the problem of describing $N(f)$ is reduced to that of describing $N(i)$, where $i : S^m \to S^n$ is the canonical inclusion map for some $m \in \mathbb{N}$.

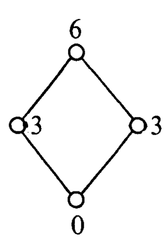
Let $f : M \to S^n$ be a map and define the span of f by $S_0(f) = S^n \cap \operatorname{span} \operatorname{im} f$, where $\operatorname{im} f$ is considered as a subset of $\mathbb{R}^{n+1}$ ($\supset S^n$).

THEOREM 5. *Let $f : M \to S^n$ be a locally rigid harmonic embedding with m-dimensional span. Then there exists a bijection $\Psi : V(f) \to V(i)$ preserving the additive relations, where $i : S^m \to S^n$ is the inclusion map. Moreover the metric spaces $N(f)$ and $N(i)$ are isometric.*

Proof. We may suppose that the span of f is $S^m \subset S^n$. If $v \in V(f)$ then local rigidity of f implies the existence of a Killing vector field $X \in so(n+1)$ such that we have $v = X \circ f$ and $(\nabla_X X) \circ f = 0$. Denote by F that (arcwise) connected component of the zero locus of $\nabla_X X$ which contains $\operatorname{im} f$. As in the proof of Theorem 2, F consists of eigenvectors of X^2, that is, $X^2 x = -\|Xx\|^2 x$ for all $x \in F$. The eigenvalues of X^2 form a discrete set and hence $\|X\|$ is a constant, say c, on F and $F \subset S^n$ on a totally geodesic submanifold, since it is the intersection of the linear subspace $\ker(X^2 - c^2 I_{m+1}) \subset \mathbb{R}^{m+1}$ with $S^m \subset F$; thus we obtain that $X \circ i \in V(i)$.

If $X \in so(n+1)$ with $v = \tilde{X} \circ f$ and $(\nabla_{\tilde{X}} \tilde{X}) \circ f = 0$ then $(X - \tilde{X}) \circ f = 0$. Since the connected components of the zero locus of a Killing vector field are totally geodesic submanifolds [**4**, Theorem 5.4, p. 62], it follows that $(X - \tilde{X}) \mid S^m = 0$, that is, $X \circ i = \tilde{X} \circ i$. Define $\Psi : V(f) \to V(i)$ by $\Psi(v) = X \circ i$, where $X \in so(n+1)$ with $X \circ f = v$ and $(\nabla_X X) \circ f = 0$. By the above reasoning, Ψ is well-defined and injective and it preserves the additive relations. Local rigidity of the inclusion map $i : S^m \to S^n$ implies that Ψ is surjective, which proves the first statement. As for the second, define $\Phi : N(f) \to N(i)$ such that, for $f' \in N(f)$, $\Phi(f')$ is the inclusion of the span $S_0(f')$ of f' into S^n. A straightforward argument shows that Φ maps $N(f)$ onto $N(i)$ isometrically; this completes the proof.

REMARK. If $f : T^2 \to S^3$ is a harmonic embedding with $e(f) = \frac{1}{2}$ then $S_0(f) = S^3$. An easy computation shows that the geometric lattice of id_{S^3} has the form [**17**]

$G(id_{S^3})$ 

and, by [17], there is no bijection $\Psi : V(f) \to V(id_{S^3})$ preserving the additive relations. Again, we obtain that f is not locally rigid.

COROLLARY. *Let $f : M \to S^n$ be a locally rigid harmonic embedding with n odd. Then there exists a contractive surjection $\theta : N(id_{S^n}) \to N(f)$.*

Proof. By the previous theorem it is enough to consider the case when $f = i : S^m \to S^n$. For $f' \in N(id_{S^n})$, define $\theta(f') = f' \circ i \in N(i)$. Clearly, the map $\theta : N(id_{S^n}) \to N(i)$ is contractive. By Theorem 3 any $v \in V(i)$ can be extended to a harmonic variation of id_{S^n} which implies that θ is surjective.

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