Acta Math. Acad. Sci. Hungar.
39 (1—3) (1982), 49—53.

HARMONIC SUBMERSIONS ONTO NONNEGATIVELY CURVED MANIFOLDS

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0. Introduction

In [3] D. TISCHLER proved that if F is an r-dimensional foliation of class C^∞ on a compact manifold V such that F is defined by pointwise linearly independent closed 1-forms $\alpha_1, \ldots, \alpha_r \in A^1(V)$ then V is the total space of a fibre bundle over the r-dimensional torus T^r. Moreover, this fibre bundle can be chosen arbitrarily close to the foliation F with respect to the C^∞-topology. The essential idea applied in the proof of Tischler's theorem was to approximate the cohomology classes of the 1-forms $\alpha_1, \ldots, \alpha_r$ by other ones with rational periods. But this proof does not give any information about the structure group of the approximating fibre bundle.

Nearly at the same time, A. LICHNEROWICZ [2] developed a powerful method for describing the intrinsic structure of compact oriented Riemannian manifolds with positive semidefinite generalized Ricci tensor field and by means of this method he proved that every Riemannian manifold M of this kind is the total space of a harmonic fibre bundle over the $b_1(M)$-dimensional flat torus such that the bundle projection is nothing but the Jacobian $J\colon M \to T^{b_1(M)}$. Comparing these two results one might expect that the fibration theorem of Lichnerowicz should be valid for arbitrary compact oriented Riemannian manifolds by choosing a possibly smaller dimensional torus as a base space and, on the other hand, the geometric construction of this fibre bundle should give a deeper insight of the structure of approximating fibre bundle occurring in Tischler's theorem, in the harmonic case. Our result, in this direction, is the following:

THEOREM A. *Let M be a compact oriented Riemannian manifold and denote q the codimension of the linear subspace I consisting of Killing vector fields X such that i_X annihilate all the harmonic 1-forms of M. Suppose that there is an r-dimensional foliation F defined by pointwise linearly independent harmonic 1-forms $\omega_1, \ldots, \omega_r \in A^1(M)$. If $r \leqq q$ then there exist harmonic 1-forms $\omega_1', \ldots, \omega_r'$ such that ω_i' is arbitrarily close to ω_i with respect to the C^0-topology, $i=1, \ldots, r$, and the 1-forms $\omega_1', \ldots, \omega_r'$ define a harmonic fibre bundle $f\colon M \to T^r$ (over the flat torus) with finite structure group.*

Investigating the case when $r=q$ we obtain the following:

THEOREM B. *Let M and q be as in Theorem* A. *Then there is a harmonic fibre bundle $f\colon M \to T^q$ with finite structure group.*

In the proofs of Theorems A and B the idea is to push together the image of the Jacobian $J\colon M \to T^{b_1(M)}$ by means of composing J with an affine surjective map of $T^{b_1(M)}$ onto a possibly less dimensional torus. This method can be applied to another situation as follows:

THEOREM C. *Let M (with base point m_0) and q be as in Theorem A and let N be a compact oriented Riemannian manifold with positive semidefinite Ricci tensor field. Suppose that there exists a harmonic surjective map $h\colon M\to N$ with the properties:*

(i) *If γ is a curve starting from m_0 such that $\int_\gamma h^*\beta=0$ for every harmonic 1-form β then γ is a loop,*

(ii) $b_1(N)\leq q$.

Then there exists a harmonic fibre bundle $f\colon M\to T^{b_1(N)}$ with finite structure group.

1. Preliminaries

Throughout this note all manifolds, maps, bundles, etc. will be smooth, i.e. of class C^∞, unless stated otherwise. We shall use the terminology of [2].

Let M be a compact oriented Riemannian manifold with metric tensor $(\,,)_M$, first Betti number $b_1(M)$ and base point m_0. Suppose that this metric tensor is extended to an inner product of tensors of any type on M. The global scalar product of p-forms $\alpha, \alpha'\in \Lambda^p(M)$ is defined by $\langle\alpha,\alpha'\rangle=\int_M(\alpha,\alpha')v$, where v denotes the volume element. The adjoint of the exterior differentiation d is denoted by ∂ and the 1-forms occurring in the kernel of the Laplacian $\Delta=d\cdot\partial+\partial\cdot d$ (i.e. the harmonic 1-forms on M) form a linear space $\mathcal{H}_M$ of dimension $b_1(M)$. Denote G_M the maximal connected subgroup of the group of isometries of M. Then G_M is a compact Lie group and its Lie algebra L_M can be identified with the Lie algebra of Killing vector fields on M. The ideal $I_M=\{X\in L_M|i_M\omega=0 \text{ for every } \omega\in\mathcal{H}_M\}$ contains the derived Lie algebra L'_M. Denote $q_M=\operatorname{codim} I_M=\dim L_M-\dim I_M$. The linear subspace $\mathcal{H}^0_M=\{\omega\in\mathcal{H}_M|i_X\omega=0$ for every $X\in L_M\}$ has codimension q_M. Let $P_M\subset\mathcal{H}^*_M$ be the discrete subgroup of maximal rank which corresponds to $H_1(M;Z)$ under the de Rham isomorphism $H_1(M;R)\cong\mathcal{H}^*_M$. If $\pi_M\colon\tilde M\to M$ denotes the universal covering, with the class $\tilde m_0\in\tilde M$ of null-homotopic loops as a base point, then there is a map $\tilde J_M\colon\tilde M\to\mathcal{H}^*_M$, defined by $\tilde J_M(\tilde m)[\omega]=u(\tilde m)-u(\tilde m_0)$, with $\tilde m\in\tilde M$, $\omega\in\mathcal{H}^*_M$ and $\pi^*_M\omega=du$. This map can be projected down yielding a map $J_M\colon M\to\mathcal{H}^*_M/P_M$ whose components with respect to flat coordinate neighbourhoods of $\mathcal{H}^*_M/P_M$ are harmonic functions. $B(M)=\mathcal{H}^*_M/P_M$ endowed with the flat metric is called the canonical torus of M and J_M is the Jacobian. The Killing vector fields on M are projected down by J_M to $B(M)$ yielding uniform vector fields, i.e. $(J_M)_*\colon L_M\to L_{B(M)}$ is a Lie algebra homomorphism and, on the group level, it induces a homomorphism $\hat J_M\colon G_M\to G_{B(M)}$, where $G_{B(M)}$ and $L_{B(M)}$ denote the group of translations of $B(M)$ and its Lie algebra, resp. Then $J_M\cdot g=\hat J_M(g)\cdot J_M$ holds for every $g\in G_M$ and the ideal I_M corresponds to the subgroup $D_M=\ker\hat J_M$.

The ideal $I_M\subset L_M$ is a direct summand and so, [4], there exists a complementary ideal $R_M\subset L_M$, with $R_M\oplus I_M=L_M$, such that the connected subgroup $Q_M\subset G_M$ which corresponds to R_M is closed. Thus Q_M is a central toroidal subgroup of G_M and $\hat J_M|Q_M\colon Q_M\to G_{B(M)}$ is a local isomorphism into G_M with finite kernel $H_M=Q_M\cap D_M$. Now let M and N be compact oriented Riemannian manifolds (with base points m_0 and n_0, resp.). A map $f\colon M\to N$ is said to be harmonic if it is an extremal of the energy functional $E(f)=\frac{1}{2}\int_M(f_*,f_*)v$, where (f_*,f_*) denotes the inner product of the vector bundle $f^*(T(N))\otimes T^*(M)$ induced by the Riemannian metrics of

M and N. If $f\colon M\to N$ is a harmonic, base point preserving map and the Ricci tensor field of N is positive semidefinite then, [4], there exists an affine map $B(f)\colon B(M)\to B(N)$ such that $B(f)\cdot J_M=J_N\cdot f$ holds. Especially, the Jacobian J_M is universal among the harmonic maps from M into flat tori.

If there is no danger of confusion the subscripts M and N will be omitted.

2. Proofs of Theorems A, B and C

A. Let M be a compact oriented Riemannian manifold, $m_0\in M$, and let $q=\operatorname{codim} I=\operatorname{codim}\mathscr{H}^0$. Consider a foliation F defined by the harmonic 1-forms $\omega_1,\dots,\omega_r\in\mathscr{H}$, with $\omega_1\wedge\dots\wedge\omega_r\neq0$ and $r\leqq q$. Then there exists $\varepsilon>0$ such that for any r-tuple of harmonic 1-forms $\bar\omega_1,\dots,\bar\omega_r\in\mathscr{H}$, with $\|\omega_i-\bar\omega_i\|<\varepsilon$, $i=1,\dots,r$, the relation $\bar\omega_1\wedge\dots\wedge\bar\omega_r\neq0$ still holds where $\|\cdot\|$ is induced by the global scalar product, i.e. the 1-forms $\bar\omega_1,\dots,\bar\omega_r$ define a foliation F ε-close to X. We state that there exist harmonic 1-forms $\omega_1',\dots,\omega_r'\in\mathscr{H}$ and a linear subspace $\mathscr{K}''\subset\mathscr{H}$ with the properties:

(1) $\|\omega_i-\omega_i'\|<\varepsilon$ for every $i=1,\dots,r$, and so $\omega_1'\wedge\dots\wedge\omega_r'\neq0$ on M,

(2) if $\mathscr{K}'\subset\mathscr{H}$ denotes the linear subspace spanned by $\omega_1',\dots,\omega_r'$ then $\mathscr{K}'\oplus\mathscr{K}''\oplus\mathscr{H}^0=\mathscr{H}$ (direct sum),

(3) using the notation $\mathscr{K}=\mathscr{K}'\oplus\mathscr{K}''\subset\mathscr{H}$, the images of P under the epimorphisms $k^*\colon\mathscr{H}^*\to\mathscr{K}^*$ and $k'^*\colon\mathscr{H}^*\to\mathscr{K}'^*$ induced by the inclusions $k\colon\mathscr{K}\subset\mathscr{H}$ and $k'\colon\mathscr{K}'\subset\mathscr{H}$, are discrete.

In order to prove this statement we consider the torus $\theta=\mathscr{H}/\Psi^{-1}(P)$, where $\Psi\colon\mathscr{H}\to\mathscr{H}^*$ is the isomorphism induced by the global scalar product on $\mathscr{H}$. The Lie algebra of θ is $\mathscr{H}$ with the natural projection $\mu\colon\mathscr{H}\to\theta$ as the exponential map.

We define the 1-forms $\omega_1',\dots,\omega_r'\in\mathscr{H}$ inductively using that $r\leqq q$ holds.

At first let $\omega_1'\in\mathscr{H}$ be such that $\|\omega_1-\omega_1'\|<\varepsilon$, $(R\omega_1')\cap\mathscr{H}^0=\{0\}$ and $\mu(R\omega_1')\subset\theta$ is closed. Assume that we defined $\omega_1',\dots,\omega_{i-1}'\in\mathscr{H}$. Then let $\omega_i'\in\mathscr{H}$ be such that $\|\omega_i-\omega_i'\|<\varepsilon$, the intersection of $R\omega_i'$ with the subspace spanned by the 1-forms $\omega_1',\dots,\omega_{i-1}'$ and $\mathscr{H}^0$ is $\{0\}$ and $\mu(R\omega_i')\subset\theta$ is closed.

Then, it is clear that (1) holds. Moreover, for the subspace $\mathscr{K}'$ spanned by $\omega_1',\dots,\omega_r'$, the relation $\mathscr{K}'\cap\mathscr{H}^0=\{0\}$ is satisfied and $\mu(\mathscr{K}')\subset\theta$ is a closed subgroup. Let $\mathscr{K}''\subset\mathscr{H}$ be a subspace such that $\mathscr{K}'\oplus\mathscr{K}''\oplus\mathscr{H}^0=\mathscr{H}$ and $\mu(\mathscr{K}'')\subset\theta$ is closed. Then (2) is also valid. Using the isomorphism $\Psi\colon\mathscr{H}\to\mathscr{H}^*$, $\Psi|\mathscr{K}\colon\mathscr{K}\to\mathscr{K}^*$ and $\Psi|\mathscr{K}'\colon\mathscr{K}'\to\mathscr{K}'^*$ the maps $k^*\colon\mathscr{H}^*\to\mathscr{K}^*$ and $k'^*\colon\mathscr{H}^*\to\mathscr{K}'^*$ will correspond to the orthogonal projections $\mathscr{H}\to\mathscr{K}$ and $\mathscr{H}\to\mathscr{K}'$, resp. Because $\mu(\mathscr{K})$ and $\mu(\mathscr{K}')$ are closed subgroups of θ, the images of $\Psi^{-1}(P)$ under these orthogonal projections are discrete, which accomplishes the proof of (3).

If $l\colon\mathscr{K}'\subset\mathscr{K}$ denotes the inclusion then $l^*\circ k^*=k'^*$ holds and the epimorphisms k^* and l^* can be projected down so that the diagram

$$\begin{array}{ccccccc}
\tilde M & \xrightarrow{\tilde J} & \mathscr{H}^* & \xrightarrow{k^*} & \mathscr{K}^* & \xrightarrow{l^*} & \mathscr{K}'^* \\
\downarrow{\scriptstyle\pi} & & \downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{J} & \mathscr{H}^*/P & \xrightarrow{k_0} & \mathscr{K}^*/k^*(P) & \xrightarrow{l_0} & \mathscr{K}'^*/k'^*(P)
\end{array}$$

commutes. Endowing $\mathcal{K}^*/k^*(P)=T^q$ and $\mathcal{K}'^*/k'^*(P)=T^r$ with the flat metrics, we state that the harmonic map $f=l_0\circ k_0\circ J$ defines a fibre bundle $f\colon M\to T^r$ with finite structure group. At first we consider the harmonic map $f_0=k_0\circ J$. If $\{\omega'_{r+1}, \dots, \omega'_q\}\subset\mathcal{K}''$ is an arbitrary base then $\omega'_1\wedge\dots\wedge\omega'_q\neq 0$ on M and the components of $\tilde{f}_0=k^*\circ\tilde{J}$ with respect to the dual base $\{\omega'^*_1, \dots, \omega'^*_q\}\subset\mathcal{K}^*$ are given by $\tilde{f}_0^i(.)=\tilde{f}_0(.)[\omega'_i]$, $i=1, \dots, q$. If u_i, $i=1, \dots, q$, is a harmonic scalar on M with $du_i=\pi^*\omega'_i$ then $\tilde{f}_0^i=u_i$ and hence $\tilde{f}_0$ has maximal rank at every point of M. Thus f_0 is a harmonic submersion onto T^q which means that $f_0\colon M\to T^q$ is a harmonic fibre bundle. Identifying the translation group $G_{B(M)}$ of $B(M)$ with $B(M)$ itself, we obtain a homomorphism $\hat{f}_0=k_0\cdot\hat{J}\colon G\to T^q$ which induces $(f_0)_*\colon L\to L_{T^q}$ on the Lie algebra level, where L_{T^q} denotes the Lie algebra of uniform vector fields on T^q. Moreover, $f_0\cdot g=\hat{f}_0(g)\cdot f_0$ holds for every $g\in G$. We assert that $\hat{f}_0|Q\colon Q\to T^q$ is a local isomorphism. To prove this, it is enough to show that $(f_0)_*|R\colon R\to L_{T^q}$ is a monomorphism. Let $0\neq X\in R$ and denote $\tilde{X}$ the lift of X to $\tilde{M}$. Choose $\omega\in\mathcal{K}$ with $i_X\omega\neq 0$ and let u be a harmonic scalar on $\tilde{M}$ with $du=\pi^*\omega$. Then, for the one-parameter group of transformations $\tilde{\varphi}$ induced by $\tilde{X}$, we have $\tilde{\varphi}_0(\tilde{m}_0)=\tilde{m}_0$ and $\dot{\tilde{\varphi}}_0(\tilde{m}_0)=\tilde{X}_{\tilde{m}_0}$. Differentiating the identity $\tilde{J}(\tilde{\varphi}_t(\tilde{m}_0))[\omega]=u(\tilde{\varphi}_t(\tilde{m}_0))-u(\tilde{m}_0)$ at $t=0$ we obtain that $\tilde{J}_*(\tilde{X})_0[\omega]=i_X\omega$, where the uniform vector field $\tilde{J}_*(\tilde{X})$ on $\mathcal{H}^*$ is identified with $\tilde{J}_*(\tilde{X})_0\in\mathcal{H}^*$. Thus $\tilde{J}_*(\tilde{X})_0|\mathcal{K}\neq 0$, i.e. $(\tilde{f}_0)_*(\tilde{X})\neq 0$ which proves that $(f_0)_*|R$ is a monomorphism.

Because $l_0\colon T^q\to T^r$ is an affine map onto T^r, it follows that $f\colon M\to T^r$ is a harmonic fibre bundle. $\hat{f}_0|Q\colon Q\to T^q$ is a local isomorphism and so there is a closed subgroup $\hat{Q}\subset Q$ such that $(l_0\cdot\hat{f}_0)|\hat{Q}\colon \hat{Q}\to T^r$ is a local isomorphism. Moreover, denoting $\hat{f}=l_0\cdot\hat{f}_0$, we have $f\cdot g=\hat{f}(g)\cdot f$ for every $g\in G$. It follows that every orbit of the action of $\hat{Q}$ on M is a finite covering space of T^r by f and hence the structure group $(\ker\hat{f})\cap\hat{Q}$ of the bundle $f\colon M\to T^r$ is finite.

B. Let $\mathcal{K}\oplus\mathcal{H}^0=\mathcal{H}$ be an arbitrary decomposition and choose a base $\{\omega_1, \dots, \omega_q\}\subset\mathcal{K}$. Then $\omega_1\wedge\dots\wedge\omega_q\neq 0$ on M and thus Theorem A implies Theorem B.

REMARK. Apart from the finiteness of the structure group, Theorems A and B are direct consequences of the Tischler's theorem. On the other hand, the finiteness of the structure group of the fibre bundle $f\colon M\to T^q$ implies that M has a finite covering space which splits diffeomorphically as the product of T^q and the typical fibre. Indeed, this splitting can be achieved by pulling back the bundle $f\colon M\to T^q$ to a principal orbit of the action of $\hat{Q}$ on M.

C. Let M and N be compact oriented Riemannian manifolds with base points m_0 and n_0, resp. and suppose that the Ricci tensor field of N is positive semidefinite. If $h\colon M\to N$ is a harmonic, base point preserving map then there exists an affine map $B(h)\colon B(M)\to B(N)$ such that $B(h)\cdot J_M=J_N\cdot h$ holds. Denote $f=B(h)\cdot J_M\colon M\to B(N)$. Identifying the translation group of $B(M)$ with $B(M)$ itself we obtain a homomorphism putting $\hat{f}=B(h)\cdot\hat{J}_M\colon G_M\to B(N)$. Moreover, $f\cdot g=\hat{f}(g)\cdot f$ holds for every $g\in G_M$. Now, assume that h is surjective and that conditions (i) and (ii) are satisfied. Because J_N is surjective [2] we obtain that $f\colon M\to B(N)$ is a surjective map, i.e. $B(h)$ maps $B(M)$ onto $B(N)$. We assert that $\operatorname{im} J_M\cap\ker B(h)$ consists of a single point (which is necessarily the base point $x_0=J_M(m_0)$ of $B(M)$). To prove this, it is enough to show that $\operatorname{im}\tilde{J}_M\cap\ker h^*\subset P_M$, where $h^*\colon \mathcal{H}^*_M\to\mathcal{H}^*_N$ is the dual map of the pullback via h, [4]. Put $\varphi\in\operatorname{im}\tilde{J}_M\cap\ker h^*$, i.e. there exists $\tilde{m}\in\tilde{M}$ with $\varphi=\tilde{J}_M(\tilde{m})$ such that $\tilde{J}_M(\tilde{m})[h^*\beta]=0$ holds for every $\beta\in\mathscr{H}_N$. Let $\gamma\in\tilde{m}$ and denote $\tilde{\gamma}$: $[0, 1]\to\tilde{M}$ the lifting of γ to $\tilde{M}$, i.e. $\tilde{\gamma}(0)=\tilde{m}_0$ and $\pi_M\cdot\tilde{\gamma}=\gamma$. If $\tilde{h}$: $\tilde{M}\to\tilde{N}$ denotes the lifting of h: $M\to N$ then $h\cdot\pi_M=\pi_N\cdot h$ holds. For every $\beta\in\mathscr{H}_N$ define the harmonic scalar v_β on $\tilde{N}$ by $v_\beta(\tilde{n}_0)=0$ and $\pi_N^*\beta=dv_\beta$. Then

$$0=\tilde{J}_m(\tilde{m})[h^*\beta]=v_\beta(\tilde{h}(\tilde{m}))=\int_{\tilde{\gamma}} d\,(v_\beta\circ\tilde{h})=\int_{\tilde{\gamma}}\tilde{h}^*\,dv_\beta=\int_{\tilde{\gamma}}\pi_M^*h^*\beta=\int_{\tilde{\gamma}}h^*\beta$$

holds. Applying (i) we obtain that $\tilde{m}$ is a homotopy class of loops which means that $\varphi=\tilde{J}(\tilde{m})=\tilde{J}(\tilde{m}_0\tilde{m})-\tilde{J}(\tilde{m}_0)\in P_M$. The relation im $J_M\cap$ ker $B(h)=\{x_0\}$ on the Lie algebra level translates into the relation im $(J_M)_*\cap$ ker $h_*=\{0\}$, where $(J_M)_*$: $L_M\to L_{B(M)}$ and h_*: $L_{B(M)}\to L_{B(N)}$ are the induced Lie algebra homomorphisms. Because dim $(\mathrm{im}(J_M)_*)\geqq$ codim $I_M=q_M$ we obtain that $\hat{f}_*$: $L_M\to L_{B(N)}$ is surjective, i.e. $\hat{f}$: $G_M\to G_{B(N)}$ is an epimorphism. It follows that $\hat{f}|Q_M$: $Q_M\to G_{B(N)}$ is a local isomorphism and hence f: $M\to B(N)$ is a harmonic fibre bundle with finite structure group $\ker\hat{f}\cap Q_M$.

References

[1] J. EELLS, L. LEMAIRE, A report on harmonic maps, *Bull. London Math. Soc.*, **10** (1978), 1—68.

[2] A. LICHNEROWICZ, Variétés Kähleriennes à premiére classe de Chern non negative et variétés Riemanniennes à courbure de Ricci généralisée non negative, *J. Differential Geometry*, **6** (1971), 47—94.

[3] D. TISCHLER, On fibering certain foliated manifolds over S^1, *Topology*, **9** (1970), 153.

[4] G. TÓTH, Fibrations of compact Riemannian manifolds (to appear).

(Received April 21, 1980)

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