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ON HARMONIC MAPS INTO LOCALLY SYMMETRIC RIEMANNIAN MANIFOLDS (*)

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1. Introduction.

The purpose of this paper is to study the global geometric properties of geodesic variations of harmonic maps. A variation is called geodesic if it is given by translating the map along geodesics defined by a prescribed vector field along this map. In Section 2 we deduce a Jacobi equation for geodesic variations and give relations between the tension fields and stress-energy tensors of maps occuring in a given variation. In this section local symmetricity of the codomain is assumed only. As immediate consequences of this treatment we obtain the celebrated Eells-Sampson homotopy theorem for flat codomains and a recent result of P. Baird and J. Eells concerning the stress energy of submersions. Supposing that the codomain is a space of constant curvature, in Section 3 we give an algebraic characterization of harmonic variations in terms of the initial data (Theorem 1) and we point out an intrinsic relation between Jacobi fields and harmonic variations (Corollary 2). The main result of this section is a rigidity theorem (Theorem 2), which is an infinitesimal version of T. Sunada's result, for positively curved codomains. In Section 4, still assuming that the codomain is a space of constant curvature, we prove that the metric space of simple harmonic maps homotopic to a given map through broken harmonic variations is a totally geodesic submanifold of the codomain (Theorem 3). A similar result has recently been obtained by R. Schoen and S. T. Yau for nonpositively curved codomains. In Section 5 we consider the case when the codomain is the complex projective space. Though an algebraic description of all harmonic variations can be given (Theorem 5)

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there are several obstructions to the existence of harmonic variations (Theorem 7). Finally, in Theorem 8, we give a geometric description of parallel harmonic variations.

Throughout this paper all manifolds, maps, bundles, etc. will be smooth, i.e. of class C^∞, unless stated otherwise. The Report [3] is our general reference for harmonic maps though we adopt the sign conventions of [8].

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2. Basic relations concerning the tension field and the stress-energy tensor.

Let (M, g) be a compact Riemannian manifold and (M', g') a complete locally symmetric ($\nabla' R' = 0$) Riemannian manifold. Given a map $f\colon M \to M'$ and a vector field v along f we define $f_t\colon M \to M'$ by $f_t = \exp' \cdot (tv)$, $t \in \boldsymbol{R}$. Let $\mathcal{F}^t = (f_t)^*(T(M'))$ be the pull-back of the tangent bundle of M' via f_t. The induced metric and connection of $\mathcal{F}^t$ will be denoted by $\langle\, , \rangle_t$ and ∇^t, resp. If $t', t'' \in \boldsymbol{R}$ then there is a canonical bundle isomorphism $\tau_{t'}^{t''}[v]\colon \mathcal{F}^{t'} \to \mathcal{F}^{t''}$ defined by the parallel transport along the geodesic segments $t \to f_t(x)$, $t \in [t', t'']$ (or $[t'', t']$) and $x \in M$. It extends to an isomorphism

$$\mathcal{F}^{t'} \otimes \Lambda^*(T^*(M)) \to \mathcal{F}^{t''} \otimes \Lambda^*(T^*(M))$$

which is also denoted by $\tau_{t'}^{t''}[v]$ and we omit 0's in f_0, $\tau_0^{t'}$, etc. We write $\tau_{t'}^{t''}$ when there is no danger of confusion.

In order to deduce a Jacobi equation for the variation of f by v, let $X_x \in T_x(M)$, $x \in M$, and choose a curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$, with $\gamma(0) = x$ and $\dot\gamma(0) = X_x$. Then $t \to \exp'(tv_{\gamma(s)})$, $t \in \boldsymbol{R}$ and $|s| < \varepsilon$, is a variation of geodesics and hence we have

$$\nabla_{d/dt} \nabla_{d/dt} \big((f_t)_* X_x\big) + R'\big((f_t)_* X_{x'} \tau_t v\big)\, \tau_t v = 0 .$$

M' is locally symmetric and so the curvature tensor R' commutes with the parallel transport. Applying τ^t to both sides of the equation above and omitting X_x we obtain

$$\text{(A)} \qquad \frac{d^2 P_v(t)}{dt^2} + R'(P_v(t), v)\, v = 0 ,$$

where $P_v(t) = \tau^t(f_t)_* \in \mathcal{C}(\mathcal{F} \otimes T^*(M))$. The two initial data for the Jacobi equation (A) are

$$\text{(B)} \qquad P_v(0) = f_* \quad \text{and} \quad \left.\frac{dP_v(t)}{dt}\right|_{t=0} = dv\,.$$

For fixed $v \in \mathcal{C}(\mathcal{F})$, equation (A) with (B) is an initial value problem with unique solution which can be expanded into a convergent power series in t.

Our purpose is to describe the behaviour of the one-parameter families of tension fields

$$\tau(f_t) = -(d^t)^*(f_t)_* = \operatorname{div}^t(f_t)_*\,, \qquad t \in \boldsymbol{R}\,,$$

[3] and stress-energy tensors

$$S(f_t) = \tfrac{1}{2}\|(f_t)_*\|^2 g - (f_t)^* g' \in \mathcal{C}(\odot^2 T^*(M))\,, \qquad t \in \boldsymbol{R}\,,$$

[1] especially to study the structure of harmonic maps occuring in the variation $t \to f_t$, $t \in \boldsymbol{R}$. Thus, in order to relate these tensors with the solution of the Jacobi equation, we have to express $\tau(f_t)$ and $S(f_t)$ in terms of the 1-forms $P_v(t)$ with values in $\mathcal{F}$.

LEMMA 1: If $w \in \mathcal{C}(\mathcal{F})$ and X is a vector field on M then

$$(\tau^t \cdot \nabla_X^t \cdot \tau_t) w - \nabla_X w = -R'\Big(\int_0^t \tau^s(f_s)_* X\, ds,\, v\Big) w\,.$$

PROOF: Straightforward, using $\nabla' R' = 0$ and the definition of the curvature tensor by parallel transport [6], p. 54.

Using Lemma 1 we have

$$\begin{aligned}\tau^t(\tau(f_t)) = \tau^t(\operatorname{div}^t(f_t)_*) &= \tau^t \operatorname{trace}\{\nabla^t(f_t)_*\} = \operatorname{trace}\{(\tau^t\nabla^t\tau_t)P_v(t)\} = \\ &= \operatorname{trace}\{\nabla P_v(t)\} - \operatorname{trace}\Big\{\Big(R'\Big(\int_0^t P_v(s)\, ds,\, v\Big)\Big)P_v(t)\Big\} = \\ &= -\Big(d^* P_v(t) + \operatorname{trace}\Big\{R'\Big(\int_0^t P_v(s)\, ds,\, v\Big) P_v(t)\Big\}\Big)\,.\end{aligned}$$

Thus, the map $f_t\colon M \to M'$ is harmonic [3] if and only if

$$\text{(C)} \qquad \Psi(v, t) = d^* P_v(t) + \operatorname{trace}\Big\{R'\Big(\int_0^t P_v(s)\, ds,\, v\Big) P_v(t)\Big\} = 0\,.$$

A vector field $v \in \mathcal{C}(\mathcal{F})$ is called *harmonic variation* if $f_t\colon M \to M'$ is harmonic for all $t \in \boldsymbol{R}$. The set of all harmonic variations of $f\colon M \to M'$ is denoted by $\mathcal{V}(f)$. Obviously, $\boldsymbol{R}\mathcal{V}(f) \subset \mathcal{V}(f)$ but, as Example 1 below shows, $\mathcal{V}(f)$ is not necessarily a linear space, i.e. $\mathcal{V}(f) + \mathcal{V}(f) \not\subset \mathcal{V}(f)$ in general. If $v \in \mathcal{C}(\mathcal{F})$ is a harmonic variation then the first variation of the energy functional E [3] yields

$$\frac{dE(f_t)}{dt} = -\int_M \langle \tau(f_t), \tau^t v \rangle_t \operatorname{vol}(M) = 0 ,$$

where $\operatorname{vol}(M)$ is the volume form of M, and hence the function $t \to E(f_t)$ must be constant.

Lemma 2: $(\operatorname{div} S(f_t))\, Z = -\langle \Psi(v, t), P_v(t)\, Z \rangle$ holds for all vector fields Z on M.

Proof: Using the identity $(\operatorname{div} S(f_t))\, Z = \langle \tau(f), f_*(Z) \rangle$, we have

$$(\operatorname{div} S(f_t))\, Z = \langle \tau(f_t), (f_t)_* Z \rangle = \langle \tau^t(\tau(f_t)), \tau^t(f_t)_* Z \rangle = -\langle \Psi(v, t), P_v(t)\, Z \rangle$$

and the result follows.

As direct consequences of our approach we obtain the following known results:

Proposition 1: If M' is flat then every map $f\colon M \to M'$ is homotopic to a harmonic map. (Eells-Sampson's homotopy theorem for flat codomain [4].)

Proof: If M' is flat, i.e. $R' = 0$, then $P_v(t) = f_* + t\, dv$ and $\Psi(v, t) = d^* f_* + t d^* dv$. The de Rham decomposition of f_* has the form $f_* = du + \Omega$, where $\Omega \in \mathcal{C}(\mathcal{F} \otimes T^*(M))$ is harmonic. Taking $v = -u$ we have $P_v(t) = (1 - t)\, du + \Omega$ and $\Psi(v, t) = (1 - t)\, d^* du$. Hence f_1 is a harmonic map homotopic to f.

Proposition 2: If $f\colon M \to M'$ is a harmonic map then the divergence of its stress-energy tensor vanishes. Moreover, if $f\colon M \to M'$ is a submersion almost everywhere and $v \in \mathcal{C}(\mathcal{F})$ such that $\operatorname{div} S(f_t) = 0$ for all $t \in R$ then v is a harmonic variation. (P. Baird-J. Eells [1].)

As Example 2 below shows the assumption that $f\colon M \to M'$ is a submersion almost everywhere cannot be dropped in the second statement.

PROPOSITION 3: If v is a harmonic variation of the map $f: M \to M'$ then

(i) $\operatorname{trace}\{R'(f_*, v) f_*\} = \nabla^2 v$,

(ii) $\operatorname{trace}\{R'(f_*, v)\, dv\} = 0$

are valid.

PROOF: Calculating the first three terms of the Taylor expansion of $\Psi(v, t)$ in t, equations (i) and (ii) are easily obtained.

We note here that equation (i) is well-known from the second variation formula of the energy functional E [3]. Vector fields $v \in \mathcal{C}(\mathcal{F})$ satisfying equation (i) are called Jacobi fields along f.

PROPOSITION 4: Suppose that M' is negatively curved. Then every nonzero harmonic variation $v \in \mathcal{C}(\mathcal{F})$ is parallel and either f is constant or f maps onto a closed geodesic γ of M' and v is tangent to γ.

PROOF: By equation (i) we have

$$0 \leqslant -\int_M \operatorname{trace}\langle R'(f_*, v)v, f_*\rangle \operatorname{vol}(M) = \int_M \langle \nabla^2 v, v\rangle \operatorname{vol}(M) \leqslant 0$$

and hence $dv = 0$ and rank $f \leqslant 1$ on M. By a theorem of J. H. Sampson [10] the result follows.

3. Harmonic maps into spaces of constant curvature.

Throughout this section M' will denote a complete Riemannian manifold of constant curvature $\sigma \neq 0$. Then the solution of the initial value problem (A) with (B) has the form

$$P_v(t) = \left\langle f_* + t\, dv, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|} + \cos(t\sqrt{\sigma}\|v\|)\left(f_* - \left\langle f^*, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}\right) + \\ + \frac{\sin(t\sqrt{\sigma}\|v\|)}{\sqrt{\sigma}\|v\|}\left(dv - \left\langle dv, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}\right)$$

(if $v_x = 0$ then we take $P_v(t) = f_* + t\, dv$ at $x \in M$).

Substituting it into (C), after a long calculation, we obtain the following expression:

$$\Psi(v, t) = \cos(\alpha t)\, d^* f_* - \frac{\sin(\alpha t)}{\alpha} \nabla^2 v + \frac{\sin(\alpha t)}{\alpha} \operatorname{trace}\{R'(f_*, v) f_*\} + \\ + \frac{2t \sin(\alpha t)}{\alpha} \operatorname{trace}\{R'(f_*, v)\, dv\} + \frac{2\sin(\alpha t) - 2\alpha t \cos(\alpha t)}{\alpha^3}.$$

$$\cdot \operatorname{trace}\{R'(dv, v)\, dv\} - \sigma \frac{\cos(\alpha t) - 1}{\alpha^2} \langle d^* f_*, v\rangle v + \sigma \frac{\sin(\alpha t) - \alpha t}{\alpha^3} \cdot$$

$$\cdot \langle \nabla^2 v, v\rangle v + \sigma \frac{\sin(2\alpha t) - 2\sin(\alpha t)}{2\alpha^3} \operatorname{trace}\{\langle R'(f_*, v) f_*, v\rangle\} v +$$

$$+ \sigma \frac{-\cos(2\alpha t) - 2\alpha t \sin(\alpha t) + 1}{\alpha^4} \operatorname{trace}\{\langle R'(f_*, v)\, dv, v\rangle\} v +$$

$$+ \sigma \frac{-\sin(2\alpha t) - 4\sin(\alpha t) + 4\alpha t \cos(\alpha t) + 2\alpha t}{2\alpha^5} \operatorname{trace}\{\langle R'(dv, v)\, dv, v\rangle\} v\,,$$

where $\alpha = \sqrt{\sigma}\|v\|$.

A complete characterization of harmonic variations into spaces of constant curvature is given as follows:

THEOREM 1: Let $f\colon M \to M'$ be harmonic. Then $v \in \mathcal{C}(\mathcal{F})$ is a harmonic variation if and only if v is a Jacobi field along f, trace $\langle f_*, dv\rangle = 0$ holds and $\|v\| = \text{const.}$ on M.

PROOF: Suppose that $v \in \mathcal{C}(\mathcal{F})$ is a harmonic variation. Calculating the coefficient of the term t^3 in the Taylor expansion of $\Psi(v, t)$ we obtain that

$$\operatorname{trace}\{R'(dv, v)\, dv\} = \sigma\langle \nabla^2 v, v\rangle v$$

is valid. Using this equality we have

$$\|v\|^2 \nabla^2(\|v\|^2) = 2\|v\|^2\langle \nabla^2 v, v\rangle + 2\|v\|^2 \operatorname{trace}\|dv\|^2 = 2 \operatorname{trace}\langle dv, v\rangle^2 \geq 0$$

and hence $\nabla^2(\|v\|^2) \geq 0$. Thus, compactness of M implies that $\|v\| = \text{const.}$ Hence $\langle dv, v\rangle = 0$ and equation (ii) of Proposition 3 implies that trace $\langle f_*, dv\rangle = 0$ holds.

Conversely, if v is a Jacobi field along f with constant norm and trace $\langle f_*, dv\rangle = 0$ is satisfied then, using the explicit expression of $\Psi(v, t)$ above, a simple calculation shows that $\Psi(v, t) = 0$ for all $t \in \boldsymbol{R}$ which completes the proof.

EXAMPLE 1: Suppose that $b_1(M) > 0$ and that the Ricci tensor field of M is positive semidefinite at every point of M. Then the Albanese map $J\colon M \to A(M)$ [9] is totally geodesic and it defines a fibre bundle over the flat Albanese torus $A(M)$ of dimension $b_1(M)$. Using the product structure of $A(M)$, let $\pi\colon A(M) \to S^1$ be a projection and denote $i\colon S^1 \to S^n$, $n \geq 2$ an isometric embedding onto a great circle of S^n. Then the composition $f = i \cdot \pi \cdot J\colon M \to S^n$ is a totally geodesic map. Let $W^1, \dots, W^{n-1}$ be orthonormal parallel sections of the normal bundle of $i: S^1 \to S^n$ and denote U the tangential field of i. Finally, choose a constant $c \in \boldsymbol{R}$ and functions $\mu_i \in \mathcal{F}(M)$, $i = 1, \ldots, n-1$, satisfying $\nabla^2 \mu_i + \operatorname{trace} \|f_*\|^2 \mu_i = 0$ and $\sum_{i=1}^{n-1} \mu_i^2 = \text{const}$.

We claim that $v = cU \cdot f + \sum_{i=1}^{n-1} \mu_i W^i \cdot f$ is a harmonic variation. By the construction above $f_* = (U \cdot f) \otimes \omega$, where ω is a 1-form on M. Thus

$$\begin{aligned}
\operatorname{trace}\{R'(f_*, v) f_*\} &= \operatorname{trace}\{\langle f_*, v\rangle f_* - \|f_*\|^2 v\} = \\
&= c \operatorname{trace} \langle f_*, U \cdot f\rangle f_* - c \operatorname{trace} \|f_*\|^2 U \cdot f - \sum_{i=1}^{n-1} \operatorname{trace} \|f_*\|^2 \mu_i W^i \cdot f = \\
&= c \operatorname{trace} (\omega \otimes \omega) U \cdot f - c \operatorname{trace} \|\omega\|^2 U \cdot f - \sum_{i=1}^{n-1} \operatorname{trace} \|f_*\|^2 \mu_i W^i \cdot f = \\
&= \sum_{i=1}^{n-1} \nabla^2 \mu_i W^i \cdot f = \nabla^2 v
\end{aligned}$$

and hence v is a Jacobi field along f. Furthermore, $\operatorname{trace} \langle f_*, dv\rangle = c\langle f_*, d(U \cdot f)\rangle = 0$ and $\|v\|^2 = c^2 + \sum_{i=1}^{n-1} \mu_i^2 = \text{const}$ which accomplishes the proof. As a special case of the example above, let $M = T^2$ (= 2-dimensional flat torus with canonical parameters $0 < \varphi < 2\pi$ and $0 < \psi < 2\pi$) and put $n = 3$. Let $f: T^2 \to S^3$ be the projection $\pi: T^2 \to S^1$, $\pi(\varphi, \psi) = \varphi$, followed by a totally geodesic embedding $i: S^1 \to S^3$.

In what follows we describe the structure of the set $\mathcal{V}(f)$ of all harmonic variations. If v is a vector field along f then

$$v = \mu_0 U \cdot f + \mu_1 W^1 \cdot f + \mu_2 W^2 \cdot f ,$$

where μ_0, μ_1 and μ_2 are scalars on T^2 considered as (2π)-periodic functions on $\boldsymbol{R}^2$ in each variable separately. By Theorem 1, an easy calculation shows that $v \in \mathcal{V}(f)$ if and only if $\Delta \mu_0 = 0$, $\Delta \mu_i = \mu_i$, $i = 1, 2$, and $\mu_0^2 + \mu_1^2 + \mu_2^2 = \text{const}$, where Δ denotes the Laplacian of T^2. Compactness of T^2 implies that $\mu_0 = \text{const}$ and moreover if $v \in \mathcal{V}(f)$ then span $\{v, U \cdot f\} \subset \mathcal{V}(f)$. Thus, it is enough to describe harmonic variations of f orthogonal to $U \cdot f$, i.e. we may put $\mu_0 = 0$. The scalars μ_1 and μ_2 are real eigenfunctions of the Laplacian with eigenvalue 1, i.e. they are first order homogeneous trigonometric polynomials in φ and ψ, [2]. A simple discussion of the possible cases will then show that condition $\mu_1^2 + \mu_2^2 = \text{const}$ implies that both μ_1 and μ_2 depend on one of the variables φ and ψ only. Thus, if μ_1

and μ_2 do not depend on ψ, we have

$$\mu_1 = a \sin\varphi - b\cos\varphi \quad \text{or} \quad \mu_1 = a\sin\varphi - b\cos\varphi\,,$$

$$\mu_2 = b\sin\varphi + a\cos\varphi \quad \text{or} \quad \mu_2 = -\,b\sin\varphi - a\cos\varphi\,,$$

where $a, b \in \boldsymbol{R}$, and analogously for the other variable. Introducing the notations

$$v^0 = U\cdot f\,,$$

$$v^1 = \sin\varphi W^1\cdot f + \cos\varphi W^2\cdot f\,,$$

$$v^2 = -\cos\varphi W^1\cdot f + \sin\varphi W^2\cdot f\,,$$

$$v^3 = \sin\varphi W^1\cdot f - \cos\varphi W^2\cdot f\,,$$

$$v^4 = -\cos\varphi W^1\cdot f - \sin\varphi W^2\cdot f\,,$$

$$v^5 = \sin\psi W^1\cdot f + \cos\psi W^2\cdot f\,,$$

$$v^6 = -\cos\psi W^1\cdot f + \sin\psi W^2\cdot f\,,$$

$$v^7 = \sin\psi W^1\cdot f - \cos\psi W^2\cdot f\,,$$

$$v^8 = -\cos\psi W^1\cdot f - \sin\psi W^2\cdot f\,,$$

we have

$$\mathcal{V}(f) = \operatorname{span}\{v^0, v^1, v^2\} \cup \operatorname{span}\{v^0, v^3, v^4\} \cup$$
$$\cup \operatorname{span}\{v^0, v^5, v^6\} \cup \operatorname{span}\{v^0, v^7, v^8\}\,,$$

i.e. $\mathcal{V}(f)$ is the union of four linear 3-spaces intersecting each other in a common line. The harmonic variations v^i, $i = 0, 1, 2, 3, 4$, have the form $v^i = Y^i\cdot f$, where Y^i is a left- or right-invariant vector field on $S^3 = \mathrm{Spin}(3)$, [6], pp. 27. The « exceptional » harmonic variations v^i, $i = 5, 6, 7, 8$, have the following property:

$$f_t = \exp'\cdot(tv^i)\colon T^2 \to S^3\,, \qquad i = 5, 6, 7, 8\,,$$

is a totally geodesic map onto a great circle of S^3 for $f \in (\pi/2)\boldsymbol{Z}$ and $f_t\colon T^2 \to S^3$ is a harmonic embedding of the torus into S^3 for $t \notin (\pi/2)\boldsymbol{Z}$. Moreover, the definition of v^i, $i = 5, 6, 7, 8$, shows that v^i is not of the form $f_*(X) + Yf$, where $X \in \mathfrak{X}(T^2)$ and $Y \in \mathfrak{X}(S^3)$ are Killing vector fields, especially the reduced nullity of f is strictly positive, [15].

As an immediate consequence of Theorem 1 we have the following:

COROLLARY 1: Let $f\colon M \to S^{2n}$ be a harmonic map with $\dim M = 2n$. If there exists a nonzero harmonic variation v then the degree of the map f is zero.

PROOF: By Theorem 1 the vector field v is a nowhere zero section of $\mathcal{F}$ and hence the Euler class $e(\mathcal{F})$ vanishes. On the other hand $0 = e(\mathcal{F}) = f^*\big(e(T(S^{2n}))\big) = \deg f e(T(S^{2n}))$ and since the Euler class of an even dimensional sphere is nonzero it follows that $\deg f = 0$.

COROLLARY 2: Let $f\colon M \to M'$ be a harmonic map. A vector field v along f is a harmonic variation if and only if v is a Jacobi field along f and $\operatorname{trace}\|(f_t)_*\|^2 = \operatorname{trace}\|f_*\|^2$ for all $t \in \boldsymbol{R}$.

PROOF: By the formula of $P_v(t)$ at the beginning of this section we have

$$\begin{aligned}\operatorname{trace}\|(f_t)_*\|^2 &= \operatorname{trace}\|P_v(t)\|^2 = \operatorname{trace}\Big\{\cos^2(\sqrt{\sigma}\|v\|t)\|f_*\|^2 + \\ &+ \sin^2(\sqrt{\sigma}\|v\|t)\Big\langle f_*, \frac{v}{\|v\|}\Big\rangle^2 + \frac{\sin^2(\sqrt{\sigma}\|v\|t)}{\sigma\|v\|^2}\|dv\|^2 + \\ &+ \Big(t^2 - \frac{\sin^2(\sqrt{\sigma}\|v\|t)}{\sigma\|v\|^2}\Big)\Big\langle dv, \frac{v}{\|v\|}\Big\rangle^2 + \frac{\sin(2\sqrt{\sigma}\|v\|t)}{\sqrt{\sigma}\|v\|}\langle f_*, dv\rangle + \\ &+ \Big(2t - \frac{\sin(2\sqrt{\sigma}\|v\|t)}{\sqrt{\sigma}\|v\|}\Big)\Big\langle f_*, \frac{v}{\|v\|}\Big\rangle\Big\langle dv, \frac{v}{\|v\|}\Big\rangle\Big\}.\end{aligned}$$

Suppose that $v \in \mathrm{C}(\mathcal{F})$ is a harmonic variation. Then, by Theorem 1, v is a Jacobi field along f and, using $\operatorname{trace}\langle f_*, dv\rangle = 0$ and $\|v\| = \text{const}$, the formula above reduces to

$$\operatorname{trace}\|P_v(t)\|^2 = \operatorname{trace}\|f_*\|^2 .$$

Conversely, suppose that v is a Jacobi field along f such that

$$\operatorname{trace}\|f_t)_*\| = \operatorname{trace}\|f_*\|^2$$

for all $t \in \boldsymbol{R}$. In view of Theorem 1 we need only to show that $\operatorname{trace}\langle f_*, dv\rangle = 0$ and $\|v\| = \text{const}$ are valid. By the formula above

$$\frac{d}{dt}\operatorname{trace}\|P_v(t)\|^2|_{t=0} = 2\operatorname{trace}\langle f_*, dv\rangle = 0 .$$

Using this and the assumption that v is a Jacobi field along f we obtain

$$\frac{d^2}{dt^2}\operatorname{trace}\|P_v(t)\|^2 = 2\operatorname{trace}\left\{\left(1-\cos\left(\sqrt{\sigma}\,\|v\|\,t\right)\right)\left\langle dv, \frac{v}{\|v\|}\right\rangle^2 + \right.$$
$$\left. + \sqrt{\sigma}\,\|v\|\sin\left(\sqrt{\sigma}\,\|v\|t\right)\left\langle f_*, \frac{v}{\|v\|}\right\rangle\left\langle dv, \frac{v}{\|v\|}\right\rangle\right\} = 0$$

and it follows easily that $\operatorname{trace}\langle dv, v\rangle^2 = 0$ is valid. By choosing an orthonormal frame $\{e^1, \ldots, e^m\} \subset T_x(M)$ at some point $x \in M$ we have $0 = \operatorname{trace}\langle dv, v\rangle^2 = \frac{1}{4}\sum_{i=1}^{m}(e^i(\|v\|^2))^2$ and hence $\|v\| = \text{const}$ on M.

COROLLARY 3: Let $f\colon M \to M'$ be a map and suppose that v is a nowhere zero vector field along f with nonconstant norm. Then there are only finitely many parameter values for which f_t is harmonic.

PROOF: Suppose that the set $T = \{t \in \boldsymbol{R} | f_t$ is harmonic$\}$ is infinite, i.e. unbounded. By the formula of $\Psi(v, t)$ above we have

$$\int_M \langle \Psi(v, t), v\rangle \operatorname{vol}(M) = t\int_M \frac{1}{\|v\|^2}\operatorname{trace}\langle dv, v\rangle^2 \operatorname{vol}(M) +$$
$$+ \text{ terms bounded in } t\,.$$

Thus $\operatorname{trace}\langle dv, v\rangle^2 = 0$ which implies that $\|v\| = \text{const}$.

Using the formula in Lemma 2 we can calculate the divergence of the stress-energy tensor of the map $f_t\colon M \to M'$ in terms of the differential f_* and $v \in \mathrm{C}(\mathcal{F})$. The explicite expression of $\operatorname{div} S(f_t)$ is rather complicated but in case when $\|v\| = \text{const} \neq 0$ it reduces to the following:

$$(-\operatorname{div} S(f_t))Z = \cos^2(\alpha t)\langle d^* f_*, f_*(Z)\rangle + \frac{\sin(\alpha t)}{\alpha}\left\{\left\langle \cos(\alpha t)\cdot\right.\right.$$
$$\cdot(-\nabla^2 v + \operatorname{trace} R'(f_*, v)f_*) + \sigma\frac{\sin(\alpha t)}{\alpha}\langle d^* f_*, v\rangle v -$$
$$-\sigma\frac{2\sin(\alpha t)}{\alpha}\operatorname{trace}\langle f_*, dv\rangle v, f_*(Z)\right\rangle + \left\langle \cos(\alpha t)\, d^* f_* + \right.$$
$$\left.\left. + \frac{\sin(\alpha t)}{\alpha}d^* f_* + \frac{\sin(\alpha t)}{\alpha}(-\nabla^2 v + \operatorname{trace} R'(f_*, v)f_*), \nabla_Z v\right\rangle\right\},$$

where Z is a vector field on M.

PROPOSITION 5: Let $f\colon M \to M'$ be a map and v be a vector field along f with $\|v\| = \text{const}$. Then $(\operatorname{div} S(f_t))Z = 0$ for all $t \in \boldsymbol{R}$ if and only if the following equations are satisfied:

$$\text{(i}'\text{)}\quad \langle d^* f_*, f_*(Z)\rangle = 0,$$

$$\text{(ii}'\text{)}\quad \langle d^* f_*, \nabla_Z v\rangle + \langle -\nabla^2 v + \operatorname{trace}\{R'(f_*, v) f_*\}, f_*(Z)\rangle = 0,$$

$$\text{(iii}'\text{)}\quad \langle -\nabla^2 v + \operatorname{trace}\{R'(f_*, v) f_*\}, \nabla_Z v\rangle + \sigma\big(\langle d^* f_*, v\rangle - 2 \operatorname{trace}\langle f_*, dv\rangle\big)\langle v, f_*(Z)\rangle = 0.$$

PROOF: Straightforward, using the formula above.

COROLLARY 4: Let $f\colon M \to M'$ be an harmonic map and v be a parallel vector field along f. Then $\operatorname{div} S(f_t) = 0$ for all $t \in \boldsymbol{R}$.

EXAMPLE 2: Let $f\colon S^1 \to S^n$ be an isometric embedding onto a great circle of S^n and let v be a parallel unit section of the normal bundle of f. Then, by Corollary 4, $\operatorname{div} S(f_t) = 0$ for all $t \in \boldsymbol{R}$ but f_t is nonharmonic for $t \notin (\pi/2)\boldsymbol{Z}$.

COROLLARY 5: Let $f\colon M \to M'$ be a map and v be a vector field along f with $\|v\| = \text{const} \neq 0$. If $\operatorname{div} S(f_{-t_0}) = \operatorname{div} S(f) = \operatorname{div} S(f_{t_0}) = 0$ for some $t_0 \notin \big(\pi/(2\sqrt{\sigma}\,\|v\|)\big)\boldsymbol{Z}$ then $\operatorname{div} S(f_t) = 0$ for all $t \in \boldsymbol{R}$.

PROOF: Let Z be a vector field on M. It is enough to show that (i')-(iii') are satisfied in Proposition 5. Equation (i') is valid because $(-\operatorname{div} S(f))Z = \langle d^* f_*, f_*(Z)\rangle = 0$. Furthermore, using (i'), we have

$$\frac{1}{2}\big\{(-\operatorname{div} S(f_{t_0}))Z - (-\operatorname{div} S(f_{-t_0}))Z\big\} = \frac{\sin(2\alpha t_0)}{2\alpha}\cdot \big\{\langle d^* f_*, \nabla_Z v\rangle + \langle -\nabla^2 v + \operatorname{trace}\{R'(f_*, v) f_*\}, f_*(Z)\rangle\big\} = 0$$

and

$$\frac{1}{2}\big\{(-\operatorname{div} S(f_{t_0}))Z + (-\operatorname{div} S(f_{-t_0}))Z\big\} = \frac{\sin^2(\alpha t_0)}{\alpha^2}\cdot \big\{\langle -\nabla^2 v + \operatorname{trace}\{R'(f_*, v) f_*\}, \nabla_Z v\rangle + \sigma\big(\langle d^* f_*, v\rangle - 2\operatorname{trace}\langle f_*, dv\rangle\big)\langle v, f_*(Z)\rangle\big\} = 0$$

which implies (ii') and (iii'), resp.

A rigidity theorem for harmonic maps into locally symmetric non-positively curved Riemannian manifold has recently been proved by T. Sunada [16]. In the case when M' is a space of constant curvature, using a result of *A.* Lichnerowicz, we obtain an infinitesimal rigidity theorem as follows:

THEOREM 2: Let $f: M \to M'$ be a harmonic Riemannian submersion almost everywhere, with M' oriented, and let V be a vector field on M. If $v = V \cdot f$ is a harmonic variation then V is a Killing vector field and for $t \in \boldsymbol{R}$ we have $f_t = h_t \cdot f$, where $h_t = \exp' \cdot (tV)$ is an isometry.

If $M = M'$ is a compact Kähler manifold and $f =$ identity then the complex analogue of this result is well-known, namely, if v is a Jacobi vector field along f then v is an infinitesimal holomorphic transformation, [12] and [15].

LEMMA 3: Let M be a totally geodesic submanifold of M' with natural inclusion $f: M \to M'$ and let $v \in \mathcal{C}(\mathcal{F})$ be a harmonic variation. By the orthogonal decomposition $v = v^{\perp} + v^{\top}$, the tangential part $v^{\top}$ is a Killing vector field on M and $v^{\perp}$ satisfies the strongly elliptic equation

$$\nabla^2 v^{\perp} + \sigma m v^{\perp} = 0 \qquad \text{where} \qquad m = \dim M .$$

PROOF: Theorem 1 implies that $\|v\| =$ const and since $M \subset M'$ is totally geodesic, the following equations hold:

$$\nabla^2 v^{\perp} + \sigma m v^{\perp} = 0 \,, \tag{1}$$

$$\nabla^2 v^{\top} + \sigma(m-1) v^{\top} = 0 \,, \tag{2}$$

$$\operatorname{trace} \langle f_*, dv^{\top} \rangle = 0 \,. \tag{3}$$

In order to prove that $v^{\top}$ is a Killing vector field on M, let β denote the 1-form on M which corresponds to $v^{\top}$ by duality. The harmonicity of f implies that trace $\langle f_*, dv^{\top} \rangle = - d^*\beta = 0$. Furthermore, by (2), we have $\nabla^2 \beta + \sigma(m-1)\beta = 0$ and so

$$\Delta\beta - 2\sigma(m-1)\beta + d(d^*\beta) = 0$$

is satisfied. By a result of A. Lichnerowicz [9] it means that $v^{\top}$ is Killing vector field, which completes the proof.

EXAMPLE 3: If $M = S^m$, $M' = S^{m+k}$ and $f: S^m \to S^{m+k}$ is the canonical embedding then, choosing an orthonormal system $w_1, \ldots, w_k$ of parallel sections of the normal bundle of f, equation (1) splits into the equations

$$\Delta \langle v^{\perp}, w_i \rangle + m \langle v^{\perp}, w_i \rangle = 0 \,, \qquad i = 1, \ldots, k \,.$$

Thus the scalar $\langle v^{\perp}, w_i\rangle$ on $S^m \subset \boldsymbol{R}^{m+1}$, being an eigenfunction of the Laplacian, is the restriction of a homogeneous linear function on $\boldsymbol{R}^{m+1}$, [2].

PROOF OF THEOREM 2: By Theorem 1, $\|V\| = \text{const}$ and hence we need only to show that V is a Killing vector field on M. Let $h_t = \exp'\cdot(tV)$, $t\in \boldsymbol{R}$. Because f is a harmonic Riemannian submersion the map h_t is harmonic [13] and [14]. Thus V is a harmonic variation of the identity of M' and so by Lemma 3, V is a Killing vector field.

If the vector field v along f is parallel then the variation of f by v can be geometrically characterized as follows:

THEOREM 3: Let $f\colon M \to S^n$ (or $\boldsymbol{R}P^n$) be a map and suppose that there exists a parallel vector field v along f with $\|v\| = 1$. Denoting by $T = \{t\in \boldsymbol{R} | f_t \text{ is harmonic}\}$ we have $\pm\pi + T\subset T$ and the following cases can occur:

(*a*) v is a harmonic variation ($T = \boldsymbol{R}$) and f is either constant of f maps onto a closed geodesic γ with v tangent to γ and the maps f_t can be obtained by rotation,

(*b*) $T\subset (\pi/2)\boldsymbol{Z}$ and the necessary and sufficient condition for $\pi/2\in T$ is that

$$\langle d^* f_*, v\rangle = 0 \qquad \text{and} \qquad \operatorname{trace}\langle f_*, v\rangle^2 v = \operatorname{trace}\langle f_*, v\rangle f_*$$

are satisfied. Moreover, $\pi/2 \in T$ implies that the Morse index of f is strictly positive, provided f is harmonic.

PROOF: Using the antipodal map of S^n we have $\pm\pi + T\subset T$. By the formula of $\Psi(v, t)$ at the beginning of this section we obtain

$$\int_M \langle \Psi(v, t), v\rangle \operatorname{vol}(M) = \frac{\sin(2t)}{2}\int_M \operatorname{trace}\{\langle f_*, v\rangle^2 - \|f_*\|^2\} \operatorname{vol}(M)\,.$$

(*a*) Suppose that there exists $t_0\in T - (\pi/2)\boldsymbol{Z}$. Then, by the equation above $\operatorname{trace}\langle f_*, v\rangle^2 = \operatorname{trace}\|f_*\|^2$ holds, i.e. $f_* = v\otimes\omega$, where ω is the 1-form on M defined by $\omega(Z) = \langle f_*(Z), v\rangle$ for all vector fields Z on M. Especially, rank $f \leqslant 1$ on M. Because v is parallel we have $df_* = dv\otimes\omega + v\otimes d\omega = 0$, i.e. $d\omega = 0$ holds. Moreover, $d^* f_* = (d^*\omega)v$ and thus $0 = \Psi(v, t_0) = (d^*\omega)v$ is valid. It follows that ω is a harmonic 1-form on M and f is harmonic. Simple calculation shows that $\Psi(v, t) = 0$ and so v is a harmonic variation. A result of J. H. Sampson [10] completes the proof of this case.

(*b*) Suppose that there exists a nonharmonic map f_t. Then $T \subset (\pi/2)\mathbf{Z}$ and it remains only to check the condition of harmonicity of $f_{\pi/2}$. But $\pi/2 \in T$ if and only if $\Psi(v, \pi/2) = \operatorname{trace} \langle d^* f_*, v\rangle v + \operatorname{trace} \langle f_*, v\rangle f_* - \operatorname{trace} \langle f_*, v\rangle^2 v = 0$. Multiplying by v, this equation splits into the two equations given in Theorem 3. If $\pi/2 \in T$ then the value of the Hessian H_f on the pair (v, v) [3] reduces to the following:

$$H_f(v, v) = \int_M \|v\|^2 \operatorname{trace} \{\langle f_*, v\rangle^2 - \|f_*\|^2\} \operatorname{vol}(M) .$$

Thus, if the Morse index of f were zero then rank $f \leq 1$ would yield a contradiction to our assumption $T \neq \mathbf{R}$.

4. Manifolds of harmonic maps.

Let M be a compact oriented Riemannian manifold and M' a complete Riemannian manifold of constant curvature $\sigma \neq 0$. Denote by Harm (M, M') the set of all harmonic maps $f\colon M \to M'$. If $f \in$ Harm (M, M') then, as in Sec. 2, define $\mathcal{V}(f) = \{v \in \mathrm{C}(\mathcal{F}) \mid v$ is a harmonic variation of $f\}$. Thus $\mathcal{V}(f)$ is contained in the finite dimensional vector space of Jacobi fields along f. By Theorem 1, if $v, v' \in \mathcal{V}(f)$ then $v + v' \in \mathcal{V}(f)$ if and only if $\langle v, v'\rangle = \text{const}$ on M. Two maps f and f' in Harm (M, M') are said to be equivalent, $f \sim f'$, if there exist maps $f^0, \ldots, f^{k+1} \in$ Harm (M, M') and vectors $v^i \in \mathcal{V}(f^i)$, $i = 0, \ldots, k$, such that $f^0 = f$, $f^{k+1} = f'$ and $\exp' \cdot v^i = f^{i+1}$, $i = 0, \ldots, k$, hold. Clearly, $\sim$ is an equivalence relation. If $f, f' \in$ Harm (M, M') are related maps then let $\varrho(f, f')$ denote the infimum of the numbers $\sum_{i=0}^{k} \|v^i\|$ for which $v^i \in \mathcal{V}(f^i)$, $\exp' \cdot v^i = f^{i+1}$, $i = 0, \ldots, k$, and $f^0 = f$, $f^{k+1} = f'$ hold.

Henceforth, let $N \subset$ Harm (M, M') be a fixed equivalence class. Clearly, ϱ is a distance function on N and thus N carries a metric space structure. An open subset $G \subset N$ is called *simple* if for any $f, f' \in G$ there exists $v \in \mathcal{V}(f)$ such that $\exp' \cdot v = f'$ holds. (In case when $M = \{\text{point}\}$ this notion reduces to the notion of simple subsets in the sense of [6] p. 159.) The equivalence class $N \subset$ Harm (M, M') is said to be *locally simple* if any harmonic map $f \in N$ has a simple neighbourhood in N.

The main result of this section is the following:

THEOREM 4: Let M be a compact oriented Riemannian manifold and let $N \subset$ Harm (M, S^n) (or Harm $(M, \mathbf{R}P^n)$) denote a locally simple equivalence class of the relation $\sim$. Then N carries a Riemannian manifold structure with the following properties:

(α) The distance function ϱ is induced by the Riemann structure of N,

(β) If $x_0 \in M$ is a base point then the evaluation map $\mathcal{E}: N \to S^n$ (or RP^n), $f \to f(x_0)$, is an isometric embedding onto a totally geodesic submanifold,

(γ) The tangent space $T_f(N)$ at $f \in N$ can be identified with $\mathcal{V}(f)$, in particular, $\mathcal{V}(f)$ is a linear space,

(δ) Geodesics of N have the form $t \to \exp' \cdot (tv)$, where $v \in \mathcal{V}(f)$, $f \in N$.

In the case when the codomain is nonpositively curved a result, similar in character to our theorem above, has been proved by R. Schoen and S. T. Yau [11]. Namely, they showed that the space of harmonic maps from M into M' which are homotopic to a given map is a compact connected totally geodesic submanifold of M'.

An immediate consequence of Theorem 4 is the following:

Corollary 4: If $f, f': M \to S^n$ (or RP^n) belong to the same locally simple equivalence class and if they agree at one point then $f = f'$.

In what follows we assume that

$$N \subset \text{Harm}\,(M, S^n) \qquad (\text{or Harm}\,(M, RP^n))$$

is a locally simple equivalence class. The proof of Theorem 4 is based on the following three lemmas:

Lemma 4: $\mathcal{V}(f)$ is a linear space for any $f \in N$.

Proof: We verify the lemma for S^n the proof being analogous for RP^n. It is enough to show that $v, v' \in \mathcal{V}(f)$ implies $\langle v, v\rangle =$ $=$ const. on M. Let $U \subset N$ be a simple neighbourhood of f and choose $\varepsilon > 0$ such that $h = \exp' \cdot (\varepsilon v)$ and $h' = \exp' \cdot (\varepsilon v')$ belong to U and $\|v\| + \|v'\| < \pi/\varepsilon$. Because U is simple there exists $w \in \mathcal{V}(h)$ with $\exp' \cdot w = h'$. Then $\varrho(h, h') \leqslant \varrho(f, h) + \varrho(f, h') \leqslant \varepsilon\|v\| + \varepsilon\|v'\| < \pi$ implies that $h(x)$ and $h'(x)$ are not antipodal points for all $x \in M$ and hence, by an appropriate choice of w, we may suppose that $\|w\| < \pi$. Thus, for fixed $x \in M$, the geodesic triangle defined by $t \to \exp' \cdot (t\varepsilon v_x)$, $t \to \exp' \cdot (t\varepsilon v'_x)$ and $t \to \exp' \cdot (tw_x)$, $0 \leqslant t \leqslant 1$, is contained in a totally geodesic 2-dimensional submanifold of S^n which can be identified with $S^2 \subset S^n$. By the law of cosines in spherical trigonometry, $\langle v_x, v'_x\rangle$ is uniquely determined by the lengths $\|v_x\|$, $\|v'_x\|$ and $\|w_x\|$ of the sides of the geodesic triangle. Theorem 1 implies that the scalars $\|v\|$, $\|v'\|$ and $\|w\|$ are constant on M and hence $\langle v, v'\rangle = \text{const}$ which accomplishes the proof.

By the previous lemma $\mathcal{V}(f)$ inherits a scalar product $\langle\, ,\rangle_f$ from the metric tensor of M'.

Moreover, if $\dim \mathcal{V}(f) = d$ and $\{v^1, \ldots, v^d\} \subset V(f)$ is an orthonormal base then the vectors $\{v^1_x, \ldots, v^d_x\} \subset T_{f(x)}(S^n)$ (or $T_{f(x)}(RP^n)$) are linearly independent for all $x \in M$. Especially, the Stiefel-Whitney classes $w_{n-d+1}(T(RP^n)), \ldots, w_n(T(RP^n))$ must be in the kernel of the homomorphism $f^*\colon H^*(RP^n, \boldsymbol{Z}_2) \to H^*(M; \boldsymbol{Z}_2)$.

Lemma 5: Let v be a harmonic variation of a map $f\colon M \to S^n$ (or $\boldsymbol{RP}^n$). Then $\tau_t[v]\colon \mathcal{V}(f) \to \mathcal{V}(f_t)$, $f_t = \exp'\cdot(tv)$, is an isomorphism of Euclidean vector spaces.

Proof: We need only to show that if $w \in \mathcal{V}(f)$ then $\tau_t[v](w) \in \in \mathcal{V}(f_t)$ holds. Without loss of generality we may assume that v and w are orthonormal. Because $\|\tau_t[v]\, w\| = \|w\| = \text{const}$ on M, by Theorem 1, we have to prove that the equations

$$\nabla^2\big(\tau_t[v](w)\big) = \operatorname{trace}\big\{R'\big((f_t)_*, \tau_t[v](w)\big)(f_t)_*\big\}\,, \tag{1}$$

$$\operatorname{trace}\big\langle (f_t)_*, d\big(\tau_t[v](w)\big)\big\rangle = 0\,, \tag{2}$$

are satisfied. At first we note that

$$\operatorname{trace}\{\langle f_*, v\rangle\langle f_*, w\rangle + \langle dv, dw\rangle\} = 0$$

is valid since

$$\begin{aligned} 0 = \nabla^2\langle v, w\rangle = \langle \nabla^2 v, w\rangle + 2\operatorname{trace}\langle dv, dw\rangle + \langle v, \nabla^2 w\rangle = \\ = 2\operatorname{trace}\{\langle R'(f_*, v) f_*, w\rangle + \langle dv, dw\rangle\} = \\ = 2\operatorname{trace}\{\langle f_*, v\rangle\langle f_*, w\rangle + \langle dv, dw\rangle\} \end{aligned}$$

holds.

In order to prove (1), write $\tau^{t'}_{t''}[v] = \tau^{t'}_{t''}$. Then, applying Lemma 1, we have

$$\begin{aligned} (\tau^t\cdot\nabla^2\cdot\tau_t)(w) &= \operatorname{trace}\{(\tau^t\cdot\nabla\cdot\tau_t)(\tau^t\cdot\nabla\cdot\tau_t)\, w\} = \\ &= \operatorname{trace}\Big\{(\tau^t\cdot\nabla\cdot\tau_t)\Big(\nabla w - R'\Big(\int_0^t P_v(s)\, ds, v\Big) w\Big)\Big\} = \\ &= \nabla^2 w - \operatorname{trace}\Big\{\nabla\Big(R'\Big(\int_0^t P_v(s)\, ds, v\Big) w\Big)\Big\} - \\ &\quad - \operatorname{trace}\Big\{R'\Big(\int_0^t P_v(s)\, ds, v\Big)\Big(\nabla w - R'\Big(\int_0^t P_v(s)\, ds, v\Big) w\Big)\Big\}\,, \end{aligned}$$

where

$$P_v(t) = \langle f_*, v\rangle v + \cos t\,(f_* - \langle f_*, v\rangle v) + \sin t\, dv\,.$$

By straightforward calculation we obtain

$$(\tau^t \cdot \nabla^2 \cdot \tau_t)(w) - \operatorname{trace}\{R'(P_v(t), w)P_v(t)\} =$$

$$= 2(1 - \cos t)\operatorname{trace}\{\langle f_*, v\rangle\langle f_*, w\rangle v - \langle dv, dw\rangle v\} = 0$$

which completes the proof of (1). Turning to the proof of (2) we have

$$\operatorname{trace}\big\langle (f_t)_*, d(\tau_t(w))\big\rangle = \operatorname{trace}\langle P_v(t), (\tau^t\cdot\nabla\cdot\tau_t)(w)\rangle =$$

$$= \operatorname{trace}\left\{\langle P_v(t), dw\rangle - \left\langle P_v(t), R\Big(\int_0^t P_v(s)\,ds, v\Big)w\right\rangle\right\} =$$

$$= \operatorname{trace}\langle P_v(t), dw\rangle + \langle \Psi(v, t), w\rangle - \langle d^* P_v(t), w\rangle =$$

$$= \operatorname{trace}\{\nabla\langle P_v(t), w\rangle\} = \operatorname{trace}\{\nabla(\cos t\,\langle f_*, w\rangle + \sin t\langle dv, w\rangle)\} =$$

$$= \sin t\,\langle \nabla^2 v, w\rangle + \sin t\operatorname{trace}\langle dv, dw\rangle =$$

$$= \sin t\operatorname{trace}\{\langle f_*, v\rangle\langle f_*, w\rangle - \langle dv, dw\rangle\} = 0$$

which completes the proof.

LEMMA 6: Let $f \in \operatorname{Harm}(M, S^n)$ (or $\operatorname{Harm}(M, RP^n)$) and let $v, w \in V(f)$ be linearly independent vectors with $\|v\|, \|w\| < 2\pi$ (or π). Then there exists $u \in V(\exp'\cdot v)$ such that $\exp\cdot u = \exp\cdot w$ and u is unique up to an integer multiple of 2π (or π).

PROOF: We verify the statement for S^n the proof being analogous for RP^n. The vectors v_x and w_x are linearly independent in $T_{f(x)}(S^n)$ for all $x \in M$ and hence they span a subbundle $\xi \subset f^*(T(S^n))$ of rank 2. Thus, for fixed $x \in M$, $\exp'(\xi_x) \subset S^n$ is a totally geodesic submanifold which can be indentified with $S^2 \subset S^n$. The linear independence of the vectors v_x and w_x implies that $\exp'(v_x)$ and $\exp'(w_x)$ are not antipodal points and so there exists a unique vector $u_x \in T_{\exp'(v_x)}(S^2)$ such that $\exp'(u_x) = \exp'(w_x)$ and $t \to \exp'(tu_x)$, $t \in [0, 1]$ is a minimising geodesic segment between $\exp'(v_x)$ and $\exp'(w_x)$. Clearly, the vectors u_x, $x \in M$, compose a vector field u along $\exp'\cdot v$ such that $\exp'\cdot u = \exp'\cdot w$ is satisfied. The linearly independent vectors $\tau_1[v]\cdot(v_x)$ and $\tau_1[v](w_x)$ are tangent to S^2 and hence there exist numbers $t_0, s_0 \in \mathbf{R}$ (mod 2π) such that $u_x = t_0\tau_1[v](v_x) + s_0\tau_1[v](w_x)$ holds. Because the scalars $\|v\|$, $\|w\|$ and $\langle v, w\rangle$ are constant on M the numbers t_0 and s_0 do not depend on the choice of x. Thus, by Lemma 5, $u \in V(\exp'\cdot v)$ which completes the proof. An immediate consequence of Lemma 6 is the following:

COROLLARY 5: Let $f, f' \in N$ with $\varrho(f, f') < \pi$ (or $\pi/2$). Then there exists a unique vector $v \in V(f)$ with $\exp'\cdot v = f'$ such that $\|v\| = \varrho\cdot \cdot(f, f')$ holds.

PROOF OF THEOREM 4: At first we consider the case

$$N \subset \mathrm{Harm}\,(M, S^n)\,.$$

Let $f \in N$ be fixed and define $\exp_f\colon V(f) \to N$ by $\exp_f(v) = \exp'\cdot v$, $v \in V(f)$. Lemma 6 easily implies that the map $\exp_f\colon V(f) \to N$ is continuous. For $a > 0$ let $B_a = \{v \in V(f) \mid \|v\| < a\}$. Then, for $0 < a < \pi$, the restriction $\exp_f |B_a \to N$ is injective. By Corollary 5, $\exp_f(B_a)$ is nothing but the open metric ball around f with radius a. It follows that $\exp_f |B_a\colon B_a \to N$ is a topological embedding onto an open neighbourhood of $f \in N$ and hence N is a topological manifold. Moreover, the family $\{(\exp_f(B_a), \exp_f^{-1}) | f \in N, 0 < a < \pi\}$ clearly compose a C^∞-atlas and so N is a differentiable manifold. (Because N is metrizable it satisfies the second axiom of countability as well.) If $v \in V(f)$, $f \in N$, then defining

$$v(\mu) = \left.\frac{d(\mu(\exp'(tv)))}{dt}\right|_{t=0},$$

where μ is a scalar on N, the vector space $V(f)$ is identified with the tangent space $T_f(N)$. Now let $x_0 \in M$ be a base point and consider the evaluation map $\mathcal{E}\colon N \in S^n$, $\mathcal{E}(f) = f(x_0)$, $f \in N$. The map $\mathcal{E}$ is clearly differentiable and $\mathcal{E}^*(v) = v_{x_0}$ is valid for $v \in V(f)$. Especially, rank $\mathcal{E} = d$ ($= \dim N$) and hence $\mathcal{E}\colon N \to S^n$ is an immersion. Let $f \in N$ be fixed. Then $\exp'\big(\mathcal{E}_*(V(f))\big) \subset S^n$ is a totally geodesic submanifold which can be identified with $S^d \subset S^n$. We claim that im $\mathcal{E} = S^d$ holds. Clearly, $S^d \subset \mathrm{im}\,\mathcal{E}$. Let $f' \in N$ and choose maps $f^0, \ldots, f^{k+1} \in \in \mathrm{Harm}\,(M, S^n)$ and vectors $v^i \in V(f^i)$, $i = 0, \ldots, k$ with $f^0 = f$ and $f^{k+1} = f'$ such that $\exp'\cdot v^i = f^{i+1}$ holds. By induction on i we prove that $f^i(x_0) \in S^d$ and $\exp'\big(\mathcal{E}_*(V(f^i))\big) \subset S^d$ are valid.

The statement is clear for $i = 0$. Suppose that $f^i(x_0) \in S^d$ and $\exp'\big(\mathcal{E}_*(V(f^i))\big) \subset S^d$ hold for some $i < k$. Then, by the induction hypothesis $f^{i+1}(x_0) = \exp'(v^i_{x_0}) = \exp'(\mathcal{E}_*(v^i)) \subset S^d$. Moreover, by Lemma 5, $\mathcal{E}_*(V(f^{i+1}))$ is the image of $\mathcal{E}_*(V(f^i))$ under the parallel transport along the geodesic segment $t \to \exp'(tv^i_{x_0})$, $t \in [0, 1]$, which lies entirely in S^d. Hence $\mathcal{E}_*(V(f^{i+1}))$ is tangent to S^d, i.e. $\exp'\big(\mathcal{E}^*(V(f^{i+1}))\big) \subset S^d$

which completes the induction step. Thus $\mathcal{E}: N \to S^d$ is a local diffeomorphism and so the manifold N inherits a Riemann structure from that of S^d such that $\mathcal{E}$ is a local isometry. With this Riemann structure the geodesics of N have the form $t \to \exp'(tv)$, where $v \in V(f)$, $f \in N$. Especially, N is a complete Riemannian manifold of constant curvature 1 and N is compact. It follows that $\mathcal{E}$: $N \to S^d$ is a Riemann covering and hence a diffeomorphism. It is clear from our construction that the distance function ϱ is induced by the Riemann structure of N, i.e. $\varrho(f, f') = r(\mathcal{E}(f), \mathcal{E}(f'))$ holds for every $f, f' \in N$, where r denotes the canonical distance function of S^d. Thus Theorem 4 is proved in the first case.

Second, consider the case $N \subset \text{Harm}\,(M, RP^n)$. Analogously to the proof above we can conclude that $\mathcal{E}$: $N \to RP^d$ is a Riemann covering. Suppose that N is a double cover of RP^d and let $f, f' \in N$ be different maps with $\mathcal{E}(f) = \mathcal{E}(f')$. Choose $v \in V(f)$ with $\exp(v) = f'$ such that γ: $[0, 1] \to N$, $\gamma(t) = \exp(tv)$, $t \in [0, 1]$, is a minimising geodesic segment connecting f and f'. Then $\mathcal{E}\cdot\gamma$ is a closed geodesic with length π, i.e. $f = f'$ which is a contradiction. Thus $\mathcal{E}$: $N \to RP^d$ is a diffeomorphism and the proof can be completed analogously to the first case.

5. **Harmonic maps into the complex projective space.**

This section is devoted to the description of harmonic variations of maps into CP^n = complex n-dimensional projective space endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4. Let f: $M \to CP^n$ be a map and $v \in \mathcal{C}(\mathcal{F})$. Using the explicite expression of the curvature tensor R' of CP^n [8] the Jacobi equation (A) of Sec. 2 becomes

$$\text{(A}')\qquad \frac{d^2P_v(t)}{dt} + \|v\|^2 P_v(t) - \langle P_v(t), v\rangle\, v + 3\langle P_v(t), \mathfrak{J}v\rangle\, \mathfrak{J}v = 0\,,$$

where $\mathfrak{J}$ denotes the complex structure of CP^n. The solution of the initial value problem (A′) with (B) has the form

$$\begin{aligned} P_v(t) = {} & \left\langle f_* + t\,dv, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|} + \cos(2\|v\|t)\left\langle f_*, \frac{\mathfrak{J}v}{\|v\|}\right\rangle \frac{\mathfrak{J}v}{\|v\|} + \\ & + \frac{\sin(2\|v\|t)}{2\|v\|}\left\langle dv, \frac{\mathfrak{J}v}{\|v\|}\right\rangle \frac{\mathfrak{J}v}{\|v\|} + \cos(\|v\|t)\left(f_* - \left\langle f_*, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|} - \right. \\ & \left. - \left\langle f_*, \frac{\mathfrak{J}v}{\|v\|}\right\rangle \frac{\mathfrak{J}v}{\|v\|}\right) + \frac{\sin(\|v\|t)}{\|v\|}\left(dv - \left\langle dv, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|} - \left\langle dv, \frac{\mathfrak{J}v}{\|v\|}\right\rangle \frac{\mathfrak{J}v}{\|v\|}\right) \end{aligned}$$

(if $v_x = 0$ then we take $P_v(t) = f_* + t\,dv$ at $x \in M$).

Our first result gives a complete description of harmonic variations of a given map $f\colon M \to CP^n$ in terms of the differential f_* and the vector fields v and $\mathfrak{J}v$ along f.

THEOREM 5: Let $f\colon M \to CP^n$ be a harmonic map. A vector field v along f is a harmonic variation if and only if $\|v\| = \text{const}$ and the following equations are satisfied:

(i) $\operatorname{trace}\{\langle f_*, v\rangle f_* - \|f_*\|^2 v + 3\langle f_*, \mathfrak{J}v\rangle \mathfrak{J}f_*\} = \nabla^2 v$

(or equivalently, v is a Jacobi field along f),

(ii) $\operatorname{trace}\{2\langle f_*, \mathfrak{J}v\rangle \mathfrak{J}\, dv + \langle f_*, \mathfrak{J}\, dv\rangle \mathfrak{J}v + \langle dv, \mathfrak{J}v\rangle \mathfrak{J}f_*\} = 0$,

(iii) $\operatorname{trace}\{2\langle f_*, \mathfrak{J}v\rangle^2 v - 2\|v\|^2\langle f_*, \mathfrak{J}v\rangle \mathfrak{J}f_* + {}$
$+ 2\langle f_*, v\rangle\langle f_*, \mathfrak{J}v\rangle \mathfrak{J}v + \langle dv, \mathfrak{J}v\rangle \mathfrak{J}\, dv\} = 0$,

(iv) $\operatorname{trace}\{\langle f_*, v\rangle\langle dv, \mathfrak{J}v\rangle + \|v\|^2\langle f_*, \mathfrak{J}\, dv\rangle\} = 0$,

(v) $\operatorname{trace}\langle f_*, \mathfrak{J}v\rangle\langle dv, \mathfrak{J}v\rangle = 0$,

(vi) $\operatorname{trace}\langle f_*, dv\rangle = 0$.

Before turning to the proof of Theorem 5 we show that it is enough to restrict ourselves to variations with constant norm.

LEMMA 7: If v is a harmonic variation of the map $f\colon M \to CP^n$ then $\|v\| = \text{const}$ on M.

PROOF: Denote $K \subset M$ the complement of the zero locus of v. Then, by a simple calculation, we have on the open set K:

$$\langle d^* P_v(t), v\rangle = t\left\langle d^*\left\{\left\langle dv, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}\right\}, v\right\rangle + \text{terms bounded in } t$$

and

$$\left\langle \operatorname{trace} R'\left(\int_0^t P_v(s)\, ds, v\right) P_v(t), v\right\rangle = \text{terms bounded in } t\,.$$

Hence, if v is a harmonic variation then $\langle \Psi(v, t), v\rangle = 0$ for all $t \in \mathbf{R}$, i.e.

$$\left\langle d^*\left\{\left\langle dv, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}\right\}, v\right\rangle = -\langle \nabla^2 v, v\rangle + \operatorname{trace}\left\{\left\langle dv, \frac{v}{\|v\|}\right\rangle^2 - \|dv\|^2\right\} = 0$$

is satisfied on K. Thus the equation

$$\|v\|^2(\langle \nabla^2 v, v\rangle + \operatorname{trace}\|dv\|^2) = \operatorname{trace}\langle dv, v\rangle^2$$

holds on M. Using this equation it follows that

$$\nabla^2(\|v\|^2) = 2(\langle \nabla^2 v, v\rangle + \text{trace}\,\|dv\|^2) \geq 0$$

is valid and compactness of M completes the proof.

COROLLARY 6: Let $f\colon M \to CP^n$ be a harmonic map with $\dim M = 2n$. If there exists a nonzero harmonic variation v then the degree of the map f is zero.

Now let v be a vector field along $f\colon M \to CP^n$ with $\|v\| = 1$. Then, by a long calculation, we obtain

$$\begin{aligned}
\Psi(v, t) = &-\sin t\, \nabla^2 v + \left(\sin t - \frac{\sin(2t)}{2}\right)\langle \nabla^2 v, \Im v\rangle \Im v + \\
&+ \text{trace}\Big\{-2\sin^2 t\langle f_*, dv\rangle v + 2\sin^2 t\langle f_*, \Im\, dv\rangle \Im v + \\
&+ 4\sin^2 t\cos t\langle f_*, \Im v\rangle \Im\, dv + 2\sin^3 t\langle dv, \Im v\rangle \Im\, dv + \\
&+ 2\sin(2t)(1-\cos t)\langle f^*, v\rangle\langle f^*, \Im v\rangle \Im v + \\
&+ \left(4\sin t\cos^2 t - \sin(4t) + \frac{\sin(2t)}{2} - \sin t\right)\langle f_*, \Im v\rangle^2 v + \\
&+ (\cos(4t) + 6\cos t\sin^2 t - \cos(2t))\langle f_*, \Im v\rangle\langle dv, \Im v\rangle v + \\
&+ \sin t(4\cos^2 t - 1)\langle f_*, \Im v\rangle \Im f_* + 2\sin^2 t(1-\cos t)\cdot \\
&\cdot\langle f_*, v\rangle\langle dv, \Im v\rangle \Im v + 2\sin^3 t(1-\cos t)\langle dv, \Im v\rangle^2 v + \\
&+ 2\sin^2 t\cos t\langle dv, \Im v\rangle \Im f_* + \sin t\langle f_*, v\rangle f_* + \\
&+ \sin t(\cos t - 1)\langle f_*, v\rangle^2 v - \frac{\sin(2t)}{2}\|f_*\|^2 + \sin t(\cos t - 1)\|dv\|^2 v\Big\}.
\end{aligned}$$

PROOF OF THEOREM 5: By Lemma 7, without loss of generality we may assume that $\|v\| = 1$ on M. Using the formula of $\Psi(v, t)$ above a simple calculation shows that equations (i)-(vi) imply $\Psi(v, t) = 0$ for all $t \in \mathbf{R}$. Instead of proving the converse statement by computing the Taylor coefficients of $\Psi(v, t)$ in t we verify a stronger assertion as follows:

THEOREM 6: Let v be a vector field along the harmonic map $f\colon M \to CP^n$ with $\|v\| = 1$ and suppose that there exists distinct numbers $t_0, s_0 \in \mathbf{R}$ with $0 < t_0, s_0 < \pi/6$ such that $f_{-t_0}, f_{t_0}, f_{-s_0}, f_{s_0}$ are harmonic. Then equations (i)-(vi) are satisfied and hence v is a harmonic variation.

PROOF: By the formula of $\Psi(v,t)$ above we have

$$\tfrac{1}{2}\langle\Psi(v,t)+\Psi(v,-t),v\rangle=$$
$$=\operatorname{trace}\{-2\sin t\langle f_*,dv\rangle+(\cos(4t)-\cos(2t))\langle f^*,\mathfrak{J}v\rangle\langle dv,\mathfrak{J}v\rangle\}\,.$$

The function $\mu\colon(0,\pi/6)\to R$ defined by

$$\mu(t)=\frac{\cos(4t)-\cos(2t)}{\sin t},\ t\in\left(0,\frac{\pi}{6}\right),$$

is strictly decreasing and so, substituting $t=t_0$ and $t=s_0$, we obtain

$$\operatorname{trace}\langle f_*,\mathfrak{J}v\rangle\langle dv,\mathfrak{J}v\rangle=\operatorname{trace}\langle f_*,dv\rangle=0\,,$$

i.e. equations (v) and (vi) are satisfied. Similarly

$$\tfrac{1}{2}\langle\Psi(v,t_0)+\Psi(v,-t_0),\mathfrak{J}v\rangle=$$
$$=2\sin^2 t_0\operatorname{trace}\{\langle f_*,v\rangle\langle dv,\mathfrak{J}v\rangle+\langle f_*,\mathfrak{J}\,dv\rangle\}=0$$

which implies equation (iv). Using these equations, we have

$$\tfrac{1}{2}(\Psi(v,t)+\Psi(v,-t))=$$
$$=2\sin^2 t\cos t\operatorname{trace}\{2\langle f_*,\mathfrak{J}v\rangle\,\mathfrak{J}\,dv+\langle f_*,\mathfrak{J}\,dv\rangle\,\mathfrak{J}v+\langle dv,\mathfrak{J}v\rangle\,\mathfrak{J}f_*\}$$

and hence (ii) is satisfied. On the other hand we have

$$\frac{1}{2}\langle\Psi(v,t)-\Psi(v,-t),v\rangle=\frac{\sin(2t)}{2}\operatorname{trace}\{2\sin^2 t(4\langle f_*,\mathfrak{J}v\rangle^2-$$
$$-\langle dv,\mathfrak{J}v\rangle^2)+\langle f_*,v\rangle^2-\|f_*\|^2-3\langle f_*,\mathfrak{J}v\rangle^2+\|dv\|^2\}$$

which yields the equations

$$\operatorname{trace}\{\langle f^*,v\rangle^2-\|f^*\|^2-3\langle f^*,\mathfrak{J}v\rangle^2+\|dv\|^2\}=0$$

and

$$\operatorname{trace}\{4\langle f^*,\mathfrak{J}v\rangle^2-\langle dv,\mathfrak{J}v\rangle^2\}=0\,.$$

Similarly

$$\frac{1}{2}\langle\Psi(v,t_0)-\Psi(v,-t_0),\mathfrak{J}v\rangle=\frac{\sin(2t_0)}{2}\cdot$$
$$\cdot\operatorname{trace}\{-\langle\nabla^2 v,\mathfrak{J}v\rangle+4\langle f_*,v\rangle\langle f_*,\mathfrak{J}v\rangle\}=0\,.$$

Using these equations, we have

$$\tfrac{1}{2}\big(\Psi(v,t)-\Psi(v,-t)\big) = -\sin t\,\nabla^2 v + \\ + \operatorname{trace}\big\{\sin t\big(\langle f_*, v\rangle f_* - \|f_*\|^2 v + 3\langle f_*, \mathfrak{J}v\rangle \mathfrak{J}f_*\big) + \\ + 2\sin^3 t\big(2\langle f_*, \mathfrak{J}v\rangle^2 v - 2\langle f_*, \mathfrak{J}v\rangle \mathfrak{J}f_* + \\ + 2\langle f_*, v\rangle\langle f_*, \mathfrak{J}v\rangle \mathfrak{J}v + \langle dv, \mathfrak{J}v\rangle \mathfrak{J}\,dv\big)\big\}$$

and hence (i) and (iii) are satisfied.

Thus Theorems 5 and 6 are proved.

In order to simplify equations (i)-(vi) of Theorem 5 we restrict ourselves to a certain class of vector fields. A vector field along f is said to be *infinitesimally real* if for every pair of linearly independent tangent vectors v_x and $\nabla_{X_x} v$ in $T_{f(x)}(CP^n)$, where $X_x \in T_x\cdot(CP^n)$, the plane ξ_x spanned by v_x and $\nabla_{X_x} v$ is real, i.e. the sectional curvature of ξ_x is 1. Obviously, v is infinitesimally real if and only if $\langle dv, \mathfrak{J}v\rangle = 0$.

COROLLARY 7: Let $f\colon M \to CP^n$ be harmonic and v be an infinitesimally real vector field along f. Then v is a harmonic variation if and only if $\|v\| = \text{const}$, v is a Jacobi field along f, $\langle f_*, \mathfrak{J}v\rangle = 0$ and $\operatorname{trace}\langle f_*, dv\rangle = 0$ are valid.

PROOF: Straightforward, reducing the system (i)-(vi) above.

Our next result gives necessary conditions for the existence of harmonic variations.

THEOREM 7: Let M be an almost Hermitian manifold and $f\colon M \to CP^n$ be a holomorphic map. If there exists an infinitesimally real harmonic variation v $(\neq 0)$ of f then $\operatorname{rank} f < 2n-2$. If f is an embedding then $\operatorname{rank} f < 2n-4$ and, moreover, if M is almost Kähler then $\operatorname{rank} f < n$.

PROOF: Corollary 7 implies that $\|v\| = \text{const.}$ and $\langle f_*, \mathfrak{J}v\rangle = 0$. Because f is holomorphic we also have $\langle f_*, v\rangle = 0$ and thus the first assertion is clear.

Assume that f is an embedding with $\operatorname{rank} f = 2n-2$. Then the normal bundle ν of f is a complex line bundle over M since v and $\mathfrak{J}v$ are nowhere zero sections of ν. As a complex hypersurface M defines a divisor and thus a complex vector bundle ξ over M. By the first adjunction formula [5] the vector bundle $\nu^* \otimes \xi$ is trivial and hence the first Chern classes $c_1(\nu)$ and $c_1(\xi)$ are equal, i.e. $c_1(\xi) = 0$. On the other hand, the fundamental class $[M] \in H_{2n-2}(CP^n; \mathbf{Z})$ is nonzero [5] and by Poincaré duality it corresponds to $c_1(\xi) = 0$ which is a contradiction.

Finally, suppose that $f: M \to CP^n$ is an embedding of the Kähler manifold M into CP^n with rank $f > n$. If v is a nonzero harmonic variation of the holomorphic map $f: M \to CP^n$ then $f_t: M \to CP^n$ is holomorphic for all $t \in \mathbf{R}$ [3]. If $|t|$ is sufficiently small then $f_t: M \to \to CP^n$ is also an embedding and since $\|v\| = \text{const.}$ the complex submanifolds $f(M)$ and $f_t(M)$ are disjoint. Indicate by α and α_t the Poincaré duals of the fundamental classes $[f(M)]$ and $[f_t(M)]$ in $H_{2m}(CP^n; \mathbf{Z})$, resp., where $m = \dim_C M$. Then $\alpha, \alpha_t \in H^{2n-2m}(CP^n; \mathbf{Z})$ are nontrivial and, by $4n - 4m < 2n$, the product $\alpha \cdot \alpha_t \in H^{4n-4m}(CP^n; \mathbf{Z})$ is also nontrivial. On the other hand, disjointness of $f(M)$ and $f_t(M)$ implies $\alpha \cdot \alpha_t = 0$ which is a contradiction.

As a final result we give a geometric description of parallel variations as follows:

THEOREM 8: Let v be a parallel vector field along a harmonic map $f: M \to CP^n$ with $\|v\| = 1$. Then the Morse index of f is strictly positive unless f is constant. Furthermore:

(*a*) If there exists $0 < t_0 < \pi/4$ such that f_{t_0} is harmonic then v is a harmonic variation of f and in this case either f is constant or f maps onto a closed geodesic γ and v, being tangent to γ, rotates the map f along γ. (Especially if M is an almost Hermitian manifold and f is holomorphic then f must be constant.)

(*b*) If $b_1(M) = 0$ and there exists $0 < t_0 < \pi/4$ such that f_{t_0} is harmonic then f is constant.

(*c*) If $\langle f_*, \mathfrak{J}v \rangle = 0$ and v is not a harmonic variation of f then

$$T = \{t \in \mathbf{R} \mid f_t \text{ is harmonic}\} \subset \frac{\pi}{2}\mathbf{Z}$$

and $\pi/2 \in T$ if and only if $\operatorname{trace} \langle f_*, v \rangle f_* = \operatorname{trace} \langle f_*, v \rangle^2 v$ holds.

PROOF: Assume that the Morse index of f is zero. Then

$$H_f(\mathfrak{J}v, \mathfrak{J}v) = -\int_M \operatorname{trace} \{\|v\|^2 \|f_*\|^2 - \langle f_*, \mathfrak{J}v \rangle^2 + 3\langle f_*, v \rangle^2\} \operatorname{vol}(M) = 0$$

implies that $\langle f_*, v \rangle = 0$ is valid. Using this equation, we have

$$H_f(v, v) = -\int_M \operatorname{trace} \{\|v\|^2 \|f_*\|^2 + 3\langle f_*, \mathfrak{J}v \rangle^2\} \operatorname{vol}(M) = 0$$

and hence f must be constant.

If v is a parallel vector field along f with $\|v\| = 1$ then the formula of $\Psi(v, t)$ reduces to the following:

$$\begin{aligned}\Psi(v, t) = \sin t \operatorname{trace} \{4 \cos t(1-\cos t)\langle f_*, v\rangle\langle f_*, \mathfrak{J}v\rangle \mathfrak{J}v + \\ + (-8\cos^3 t + 4\cos^2 t + 5\cos t - 1)\langle f_*, \mathfrak{J}v\rangle^2 v + \\ + (4\cos^2 t - 1)\langle f_*, \mathfrak{J}v\rangle \mathfrak{J}f_* + \langle f_*, v\rangle f_* + \\ + (\cos t - 1)\langle f_*, v\rangle^2 v - \cos t \|f_*\|^2 v\}\,.\end{aligned}$$

(a) Suppose that f_{t_0} is harmonic for some $0 < t < \pi/2$. By the formula of $\Psi(v, t)$ above we have

$$\begin{aligned}\int_M \langle \Psi(v, t_0), v\rangle \operatorname{vol}(M) = \frac{\sin(2t_0)}{2}\left(\int_M \operatorname{trace}\{\langle f_*, v\rangle^2 + \langle f_*, \mathfrak{J}v\rangle^2 - \|f_*\|^2\}\cdot\right.\\ \left.\cdot \operatorname{vol}(M) - 4\cos(2t_0)\int_M \operatorname{trace}\langle f^*, \mathfrak{J}v\rangle^2 \operatorname{vol}(M)\right) = 0\end{aligned}$$

and hence $\langle f_*, \mathfrak{J}v\rangle = 0$ and trace $\{\langle f_*, v\rangle^2 - \|f_*\|^2\} = 0$. By the last equality, $f_* = v \otimes \omega$, where ω is a harmonic 1-form on M. It follows that $\Psi(v, t) = 0$ for all $t \in \boldsymbol{R}$, i.e. v is a harmonic variation. The rest of case (a) follows from a result of J. H. Sampson [10].

(b) By the previous case we may assume that $f_{\pi/4}$ is harmonic. Then

$$\left\langle \Psi\left(v, \frac{\pi}{4}\right), v\right\rangle = \frac{1}{2}\operatorname{trace}\{\langle f_*, v\rangle^2 + \langle f_*, \mathfrak{J}v\rangle^2 - \|f_*\|^2\} = 0$$

which implies that $f_* = v \otimes \omega + \mathfrak{J}v \otimes \omega'$, where ω and ω' are harmonic 1-forms on M. Thus $b_1(M) = 0$ implies that f is constant.

(c) If f_{t_0} is harmonic for some $t_0 \notin (\pi/2)\boldsymbol{Z}$ then the equation $\langle \Psi(v, t_0), v\rangle = 0$ implies that v is a harmonic variation. Thus $T \subset$ $\subset (\pi/2)\boldsymbol{Z}$ and the rest is clear.

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