On Variations of Harmonic Maps into Spaces of Constant Curvature (*).

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Summary. – *This paper studies the global geometric properties of geodesic variations of harmonic maps into spaces of constant curvature, i.e. variations given by translating the maps along geodesics defined by prescribed vector fields. Description of variations through harmonic maps is given and an infinitesimal rigidity of harmonic maps is shown.*

A map $f\colon M \to M'$ of a compact and oriented Riemannian manifold M into a complete Riemannian manifold M' induces a specific 1-form f_* on M with values in the pull-back bundle $\mathcal{F} = f^*(T(M'))$, i.e. $f_* \in C^\infty(\mathcal{F} \otimes T^*(M))$. The map f is said to be harmonic, [1], if it is an extremal of the energy functional

$$E(f) = \frac{1}{2}\int_M \|f_*\|^2 \operatorname{vol}(M) .$$

The Euler-Lagrange equation associated to the energy functional E has the form $\tau(f) = 0$, where $\tau(f) \in C^\infty(\mathcal{F})$ denotes the trace of the second fundamental form of f. By developing a Laplace operator Δ on the Riemannian-connected bundle $\mathcal{F} \otimes \Lambda^*(T^*(M))$ the map f is harmonic if and only if $\Delta f_* = 0$, or equivalently $\partial f_* = 0$, where ∂ denotes the adjoint of the exterior differentiation d on $\mathcal{F} \otimes \Lambda^*(T^*(M))$.

By a famous theorem of J. Eells and J. H. Sampson, [2], if M' is nonpositively curved then the heat equation

$$\frac{\partial f_t}{\partial t} = \tau(f_t) ,$$

with $f_0 = f$, deforms an arbitrary map into a harmonic one. The problem of deforming $f\colon M \to M'$ into a harmonic map, in case when M' is positively curved, proved to be extremely difficult. Although a good many particular results are known, mainly for low dimensional manifolds, one of the best-known results was achieved by R. T. SMITH, [8] and [9], by showing that every map between Euclidean n-spheres can be deformed into a harmonic one, for $n \leq 7$.

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A vector field v along f defines a homotopy of f by

$$f_t(x) = \exp' (tv_x) , \qquad x \in M, \ t \in R .$$

This homotpy is called (geodesic) variation of f by v.

The subject of this paper is to study variations of f through harmonic maps. In Section 1 we deduce two equations with two initial data which describe the variation of $f: M \to M'$ by v. The first equation is equivalent to the harmonicity of f_t and the second one, with the two initial data, yields an initial value problem for a Jacobi equation. If M' is locally symmetric and negatively curved then, taking the first two terms of the Taylor expansion of the solution, we obtain a result of P. HARTMAN, [3] and [7], as follows:

A nonzero vector field v along f defines a variation of f through harmonic maps if and only if $dv = 0$ and either f is constant or f maps onto a closed geodesic γ of M' and v is tangent to γ.

In Section 2 we deal with the case when M' is a space of constant curvature σ. Then the initial value problem can be solved explicitly and it yields that if f is nonharmonic and v is nowhere zero and of nonconstant norm then f_t is harmonic only for finitely many values of t.

Using a result of A. Lichnérowicz, we obtain that if $f: M \subset M'$ is a totally geodesic submanifold of M' and v is a nowhere zero vector field along f which defines a harmonic variation of f then the projection $v^\top$ of v onto $T(M)$ is a Killing vector field on M and $v^\perp = v - v^\top$ is a solution of a strongly elliptic equation.

Especially, it follows that the nowhere zero harmonic variations of the identities of the Euclidean spheres consist of isometries.

As another application we obtain the Eells-Sampson's homotopy theorem for flat target manifolds.

Throughout this paper, all manifolds, maps, bundles, etc. will be smooth, i.e. of class C^∞ and [1] will be our general reference for the notions and notations used here.

1. – Equations for variations.

From now on, let M be a compact and oriented Riemannian manifold and M' be a complete and locally symmetric Riemannian manifold. Given a map $f: M \to M'$ and a vector field v along f we define $F: M \times R_+ \to M'$ and $f_t: M \to M'$, $t \geqq 0$, by $F(x, t) = f_t(x) = \exp' (tv_x)$, $x \in M$. (In what follows, we shall restrict ourselves to nonnegative values of t.)

Let $\mathcal{F}^t = (f_t)^*(T(M'))$ be the pull-back of the tangent bundle of M' via f_t. The canonical connection and metric of $\mathcal{F}^t$ will be denoted by ∇^t and $\langle\, , \rangle_t$, resp. If $0 \leqq t' \leqq t''$ then there is a canonical bundle isomorphism $\tau^{t'}_{t''}: \mathcal{F}^{t'} \to \mathcal{F}^{t''}$ induced by the parallel transport along the geodesic segments $t \to f_t(x)$, $t' \leqq t \leqq t''$ and $x \in M$.

The canonical extension

$$\tau_{t''}^{t'} \otimes id: \mathcal{F}^{t'} \otimes \Lambda^*(T^*(M)) \to \mathcal{F}^{t''} \otimes \Lambda^*(T^*(M))$$

will also be denoted by $\tau_{t''}^{t'}$ and we omit 0's in f_0, $\tau_{t''}^{0}$, etc.

Note that $v \in C^\infty(\mathcal{F})$ and $f_* \in C^\infty(\mathcal{F} \otimes T^*(M))$.

LEMMA 1. – If w is a vector field along f and X is a vector field on M then

$$((\tau_t)^{-1} \circ \nabla_X^t \circ \tau_t)(w) - \nabla_X w = -R'\Big(\int_0^t (\tau_s)^{-1}(f_s)_* X\, ds, v\Big) w$$

holds for $t \geqq 0$. (We adopt the sign convention used by K. NOMIZU.)

PROOF. – If $\mu: [a, b] \to M'$ is a curve then denote

$$\tau(\mu): T_{\mu(a)}(M') \to T_{\mu(b)}(M')$$

the parallel transport along μ. Let $x \in M$ and choose a curve $\gamma: [-\varepsilon, \varepsilon] \to M$ with $\gamma(0) = x$ and $\dot\gamma(0) = X_x \in T_x(M)$. Then γ defines a one-parameter family of curves $\delta_t: [-\varepsilon, \varepsilon] \to M'$, $t \geqq 0$, by $\delta_t(s) = f_t(\gamma(s))$, $|s| \leqq \varepsilon$. Define

$$Q: R_+ \to T_{f(x)}(M')$$

by

$$Q(t) = \lim_{r \to 0} \frac{1}{r} \{(\tau_t)^{-1} \circ \tau(\delta_t|[0, r])^{-1} \circ \tau_t \circ \tau(\delta_0|[0, r])(w_x) - w_x\} = \\ = ((\tau_t)^{-1} \circ \nabla_{X_x}^t \circ \tau_t) w - \nabla_{X_x} w .$$

Then

$$\frac{dQ(t)}{dt} = -(\tau_t)^{-1} R'((f_t)_* X_x, \tau_t(v_x))\, \tau_t(w_x) = -R'((\tau_t)^{-1}(f_t)_* X_x, v_x) w_x$$

which accomplishes the proof.

Using the formula, [10],

$$\partial^t \varphi = -\operatorname{trace}\{(X, Y) \to \nabla_X^t \tau_Y \varphi\}, \qquad \varphi \in C^\infty(\mathcal{F}^t \otimes \Lambda^*(T^*(M))),$$

we have

$$\partial^t (f_t)_* = (\tau_t \circ \partial \circ \tau_t^{-1})(f_t)_* + \tau_t \operatorname{trace}\Big\{(X, Y) \to R'\Big(\int_0^t (\tau_s)^{-1}(f_s)_* X\, ds, v\Big)(\tau_t)^{-1}(f_t)_* Y\Big\}.$$

The map f_t is harmonic if and only if $\partial^t(f_t)_* = 0$, i.e. if

$$\text{(A)}\qquad \Psi(v,t) = \partial P_v(t) + \operatorname{trace}\left\{(X, Y) \to R'\Big(\int_0^t P_v(s) X\, ds, v\Big) P_v(t) Y\right\} = 0\,,$$

where $P_v(t) = (\tau_t)^{-1}(f_t)_* \in C^\infty(\mathcal{F} \otimes T^*(M))$, $t \geqq 0$.

On the other hand, using the notations of the proof of Lemma 1, $H = F \circ (\gamma \times id)$ is a geodesic variation and hence, [5], $\partial H(s,t)/\partial s|_{s=0} = (f_t)_* X_x$ is a Jacobi field along the geodesic $t \to \exp'(tv_x)$. Thus we have

$$\nabla_{d/dt}\nabla_{d/dt}\big((f_t)_* X\big) + R'\big((f_t)_* X, \tau_t(v)\big)\tau_t(v) = 0\,.$$

Transforming both sides into $C^\infty(\mathcal{F})$ by $(\tau_t)^{-1}$ we get

$$\text{(B)}\qquad \frac{d^2 P_v(t) X}{dt^2} = -R'\big(P_v(t)X, v\big)v\,,$$

where X is a vector field on M and $t \geqq 0$.

LEMMA 2. – $\nabla_{d/dt|_{t=0}}\big(\exp'\circ(tv)\big)_* X = \nabla_X v$.

PROOF. – Let $e\colon M \to R^N$ be an isometric embedding and denote $i\colon T(R^N) \to T_0(R^N) = R^N$ and $\tilde{\nabla}$ the identification map and the standard connection of R^N, resp. Let $x \in M$ and choose a curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = X_x$.

$$i\Big(\tilde{\nabla}_{d/dt|_{t=0}}\big(\exp'\circ(tv)\big)_* X_x\Big) = \frac{d}{dt}\Big|_{t=0}\Big(i\big(\exp'\circ(tv)\big)_* X_x\Big) =$$
$$= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\exp'(tv_{\gamma(s)}) = \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0}\big(\exp'(tv_{\gamma(s)})\big) =$$
$$= \frac{d}{ds}\Big|_{s=0} i(v_{\gamma(s)}) = i\big(\tilde{\nabla}_{d/ds|_{s=0}} v_{\gamma(s)}\big)\,.$$

and hence

$$e_*\nabla_{d/dt|_{t=0}}\big(\exp'\circ(tv)\big)_* X_x = \Big(\tilde{\nabla}_{d/dt|_{t=0}} e_*\big((\exp'\circ(tv))_* X_x\big)\Big)^T = \big(\tilde{\nabla}_{d/ds|_{s=0}}(e_* v_{\gamma(s)})\big)^T = e_*\nabla_{X_x} v\,,$$

where $\top$ denotes projection to $T(e(M'))$. Thus the lemma is proved.

By the previous lemma the initial conditions for P_v are

$$\text{(C)}\qquad P_v(0) = f_* \quad\text{and}\quad \frac{dP_v(t)}{dt}\Big|_{t=0} = dv\,.$$

For fixed v, equation (B) with (C) is an initial value problem with unique solution.

REMARK. – Let $f\colon M \to M'$ be a map and consider a variation $f_t = \exp' \circ (tv)$ defined by a vector field along f. The initial value problem (B) and (C) yields

$$\frac{dP_v(t)X}{dt} = \nabla_X v - R'\left(\int_0^t P_v(s)X\,ds, v\right)v\,.$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}\|P_v(t)\|^2 = \left\langle \frac{dP_v(t)}{dt}, P_v(t)\right\rangle = \langle dv, P_v(t)\rangle - \left\langle R'\left(\int_0^t P_v(s)\;ds, v,\right)v, P_v(t)\right\rangle = \langle dv, P_v(t)\rangle + \left\langle \text{trace}\, R'\left(\int_0^t P_v(s)\,ds, v\right)P_v(t), v\right\rangle = \langle dv, P_v(t)\rangle + \langle \Psi(v,t), v\rangle - \langle \partial P_v(t), v\rangle$$

and so

$$\langle \Psi(v,t), v\rangle = \frac{1}{2}\frac{d}{dt}(\|P_v(t)\|^2) + t(-\langle \nabla^2 v, v\rangle - \|dv\|^2) + \langle \partial f_*, v\rangle - \langle f_*, dv\rangle\,.$$

Especially, integrating over M we obtain the first variation of the energy functional

$$\frac{dE(f_t)}{dt} = \frac{1}{2}\int_M \|P_v(t)\|^2 \operatorname{vol}(M) = \int_M \langle \Psi(v,t), v\rangle \operatorname{vol}(M) = \int_M \langle \partial^t (f_t)_*, \tau_t(v)\rangle \operatorname{vol}(M)\,.$$

EXAMPLE 1. If M' is flat then the de Rham decomposition of $f_* \in C^\infty(\mathcal{F} \otimes T^*(M))$ has the form $f_* = du + \Omega$, where Ω is harmonic. Hence

$$v = -u \quad \text{and} \quad P_v(t) = d(1-t)u + \Omega$$

are the solutions of $\Psi(v, 1) = 0$ and (B) with (C).

On the other hand, $\Psi(v,t) = \partial f_* - \nabla^2 vt$, where $\nabla^2 v = \text{trace}\,\{(X, Y) \to \nabla_X \nabla_Y v\}$, [6]. Thus we obtain the following

PROPOSITION 1. – If M' is flat then every map $f\colon M \to M'$ is homotopic to a harmonic one. A vector field v along f defines a variation of f through harmonic maps if and only if f is harmonic and $dv = 0$.

THEOREM 1. – Let $f\colon M \to M'$ be a harmonic map and suppose that a vector field v along f defines a variation of f through harmonic maps. Then

(1) $\text{trace}\, R'(f_*, v) f_* = \nabla^2 v$,

(2) $\text{trace}\, R'(f_*, v)\, dv = 0$.

PROOF. – If v defines a variation of f through harmonic maps then $\Psi(v, t) = 0$, especially

$$\left.\frac{d\Psi(v, t)}{dt}\right|_{t=0} = \left.\frac{d^2\Psi(v, t)}{dt^2}\right|_{t=0} = 0 .$$

From equation (B) with (C) we have

$$P_v(0) = f_* , \quad \left.\frac{dP_v(t)}{dt}\right|_{t=0} = dv \quad \text{and} \quad \left.\frac{d^2P_v(t)}{dt^2}\right|_{t=0} = -R'(f_*, v)v .$$

Now, a simple calculation yields equations (1) and (2).

REMARK. – Equation (1) can also be obtained by studying the second variation of the energy functional E, [1].

Now suppose that M' is negatively curved, i.e. all the sectional curvatures of M are negative. Using equation (1) we have

$$0 \geqq \int_M \langle \nabla^2 v, v\rangle \operatorname{vol}(M) = -\int_M \operatorname{trace}\langle R'(f_*, v)v, f_*\rangle \operatorname{vol}(M) \geqq 0$$

and hence $dv = 0$. If $v \neq 0$ then $\|v\| = \text{const.} \neq 0$ and so rank $f \leqq 1$ on M. By a theorem of J. H. Sampson, [7], either f is constant or f maps onto a closed geodesic γ of M'. Thus we have

PROPOSITION 2. – Let $f\colon M \to M'$ be a harmonic map, where M' is a negatively curved Riemannian manifold. If the vector field v along f is not identically zero and if v defines a variation of f through harmonic maps then $dv = 0$ and either f is constant or f maps onto a closed geodesic γ of M' and v is tangent to γ.

2. – Harmonic maps into spaces of constant curvature.

Throughout this section M' will denote a complete manifold of constant curvature $\sigma \neq 0$.

Then equations (A) and (B) with initial conditions (C) have the forms

$$(\mathrm{A}_\sigma) \qquad \Psi(v, t) = \partial P_v(t) + \sigma \operatorname{trace}\left\{(X, Y) \to \langle P_v(t)X, v\rangle \int_0^t P_v(s)Y\,ds - \left\langle P_v(t)X, \int_0^t P_v(s)Y\,ds\right\rangle v\right\} = 0$$

and

$$(\mathrm{B}_\sigma) \qquad \frac{d^2P_v(t)X}{dt^2} = -\sigma\|v\|^2 P_v(t)X + \sigma\langle P_v(t)X, v\rangle v$$

with

$$(\mathrm{C}_\sigma) \qquad P_v(0) = f_* \quad \text{and} \quad \left.\frac{dP_v(t)}{dt}\right|_{t=0} = dv\,.$$

LEMMA 3. – Let P_0, $P_1 \in C^\infty(\mathcal{F} \otimes T^*(M))$ and $v \in C^\infty(\mathcal{F})$. Then the solution of equation (B_σ) with initial conditions

$$(\mathrm{C}'_\sigma) \qquad P_v(0) = P_0 \quad \text{and} \quad \left.\frac{dP_v(t)}{dt}\right|_{t=0} = P_1$$

has the form

$$\begin{aligned} P_v(t)X_x = \left\langle P_0(X_x) + tP_1(X_x), \frac{v_x}{\|v_x\|}\right\rangle \frac{v_x}{\|v_x\|} + \cos\left(t\sqrt{\sigma}\|v_x\|\right)\cdot \\ \cdot\left(P_0(X_x) - \left\langle P_0(X_x), \frac{v_x}{\|v_x\|}\right\rangle \frac{v_x}{\|v_x\|}\right) + \\ + \frac{\sin\left(t\sqrt{\sigma}\|v_x\|\right)}{\sqrt{\sigma}\|v_x\|}\left(P_1(X_x) - \left\langle P_1(X_x), \frac{v_x}{\|v_x\|}\right\rangle \frac{v_x}{\|v_x\|}\right), \qquad \text{if } v_x \neq 0, \end{aligned}$$

and

$$P_v(t)X_x = P_0(X_x) + tP_1(X_x)\,, \qquad \text{if } v_x = 0\,,$$

where $X_x \in T_x(M)$.

PROOF. – Simple calculation.

REMARK. – If $\sigma < 0$ then the formula above makes sense because of the relations

$$\sin(iz) = i\sinh(z) \quad \text{and} \quad \cos(iz) = \cosh(z)\,,$$

where $i \in \boldsymbol{C}$ is the complex unit.

Taking $P_0 = f_*$ and $P_1 = dv$ in the solution of (B_σ) and substituting it into the expression of $\Psi(v, t)$ we obtain the following formula

$$\begin{aligned} \Psi(v, t) = \cos(\alpha t)\,\partial f_* - \frac{\sin(\alpha t)}{\alpha}\nabla^2 v + \frac{\sin(\alpha t)}{\alpha}\operatorname{trace} R'(f_*, v)\,f_* + \\ + \frac{2t\sin(\alpha t)}{\alpha}\operatorname{trace} R'(f_*, v)\,dv + \frac{2\sin(\alpha t) - 2\alpha t\cos(\alpha t)}{\alpha^3}\operatorname{trace} R'(dv, v)\,dv - \\ - \sigma\frac{\cos(\alpha t) - 1}{\alpha^2}\langle\partial f_*, v\rangle v + \sigma\frac{\sin(\alpha t) - \alpha t}{\alpha^3}\langle\nabla^2 v, v\rangle v + \\ + \frac{\sin(2\alpha t) - 2\sin(\alpha t)}{2\alpha\|v\|^2}\operatorname{trace}\langle R'(f_*, v)\,f_*, v\rangle v + \frac{-\cos(2\alpha t) - 2\alpha t\sin(\alpha t) + 1}{\alpha^2\|v\|^2}\cdot \end{aligned}$$

$$\cdot \operatorname{trace} \langle R'(f_*, v)\, dv, v\rangle v + \frac{-\sin(2\alpha t) - 4\sin(\alpha t) + 4\alpha t\cos(\alpha t) + 2\alpha t}{2\alpha^3 \|v\|^2} \cdot$$
$$\cdot \operatorname{trace} \langle R'(dv, v)\, dv, v\rangle v\,,$$

where $\alpha = \sqrt{\sigma}\|v\|$ and if $v_x = 0$ at $x \in M$ then we take the corresponding limits.

In case when M' is a space of constant curvature Theorem 1 can be sharpened as follows

THEOREM 2. – Let $f\colon M \to M'$ be a harmonic map and v be a vector field along f. Then v defines a variation of f through harmonic maps if and only if the following equations are valid:

(1) $\operatorname{trace} R'(f_*, v) f_* = \nabla^2 v$,

(2) $\operatorname{trace} R'(f_*, v)\, dv = 0$,

(3) $\operatorname{trace} R'(dv, v)\, dv = \sigma \langle \nabla^2 v, v\rangle v$.

PROOF. – Taking the Taylor expansion of $\Psi(v, t) = 0$ in t up to the fourth degree, by a simple calculation, equations (1)-(2)-(3) can be obtained. Substituting these equations into the expression of $\Psi(v, t)$ above we get that $\Psi(v, t)$ is identically zero.

THEOREM 3. – Let $f\colon M \to M'$ be a map and suppose that v is a nowhere zero vector field along f for which $\|v\|$ is not identically constant on M. Then there are only finitely many parameter values $t \geq 0$ for which $f_t = \exp' \circ (tv)$ is harmonic.

PROOF. – Suppose that the set $\{t \geq 0 | f_t \text{ is harmonic}\}$ is infinite. Then

$$\int_M \langle \Psi(v, t), v\rangle \operatorname{vol}(M) = -t\int_M \langle \nabla^2 v, v\rangle \operatorname{vol}(M) - t\int_M \langle dv, dv\rangle \operatorname{vol}(M) + $$
$$+ t\int_M \frac{1}{\|v\|^2} \operatorname{trace} \langle dv, v\rangle^2 \operatorname{vol}(M) + O(1)$$

holds, i.e. $\int_M 1/\|v\|^2 \operatorname{trace} \langle dv, v\rangle^2 \operatorname{vol}(M) = 0$. Thus $\operatorname{trace} \langle dv, v\rangle^2 = 0$ identically on M and choosing an orthonormal frame $\{e_i\} \subset T_x(M)$, $x \in M$, we obtain

$$\operatorname{trace} \langle dv, v\rangle^2 = \frac{1}{4} \sum_i \left(e_i(\|v\|^2)\right)^2 = 0\,,$$

i.e. $\|v\|$ is constant which accomplishes the proof.

REMARK. – If $\|v\|$ is constant on M then $\Psi(v, t)$ is bounded in t.

COROLLARY 1. – Let $f\colon M \to M'$ be a harmonic map and v be a nowhere zero vector field along f. Then v defines a variation of f through harmonic maps if and only if $\|v\|$ is constant on M and

$$\operatorname{trace} R'(f_*, v)f_* = \nabla^2 v \quad \text{and} \quad \langle f_*, dv\rangle = 0$$

are satisfied.

Now suppose that $f\colon M \subset M'$ is a totally geodesic submanifold and let v be a nowhere zero vector field along f which defines a variation of f through harmonic maps.

There is an orthogonal decomposition $v = v^\perp + v^\top$, where $v^\perp$ is orthogonal to M and $v^\top$ is tangent to M. Because M is a totally geodesic submanifold of M', Corollary 1 yields the following equations

$$\nabla^2 v^\perp + \sigma m v^\perp = 0\,, \quad \nabla^2 v^\top + \sigma(m-1)v^\top = 0 \quad \text{and} \quad \langle f_*, dv^\top\rangle = 0\,,$$

where $m = \dim M$. The first equation for $v^\perp$ is strongly elliptic and has uniqueness in the Cauchy problem, [6], i.e. if $v^\perp|U = 0$ where $U \subset M$ is an open set, then $v^\perp = 0$ identically on M.

In what follows we shall rewrite the equations for $v^\top$. Denote α the 1-form of M which corresponds to $v^\top$ by duality. Because f is harmonic

$$\langle f_*, dv^\top\rangle = -\,\partial\alpha = 0\,.$$

Furthermore, $\nabla^2\alpha + \sigma(m-1)\alpha = 0$ holds and so α satisfies the equation

$$\Delta\alpha - 2\sigma(m-1)\alpha + d\,\partial\alpha = 0\,.$$

By a result of A. Lichnérowicz, [4], it means that $v^\top$ is a Killing vector field. Thus we obtain the following

THEOREM 4. – Let $f\colon M \subset M'$ be a totally geodesic submanifold of M' and let v be a nowhere zero vector field along f which defines a variation of f through harmonic maps. By the orthogonal decomposition $v = v^\perp + v^\top$ the vector field $v^\top$ along f satisfies the strongly elliptic equation

$$\nabla^2 v^\perp + \sigma m v^\perp = 0\,,$$

where $m = \dim M$, and $v^\top$ is a Killing vector field on M.

If v is a nowhere zero vector field on S^m then v can be considered as a vector field along the identity. If v defines a variation of the identity through harmonic maps then $\|v\|$ is constant and v is a Killing vector field on S^m. Thus the integral curves of v are geodesics and so $f_t = \exp' \circ (tv)$ is an isometry for every $t \geqq 0$.

Now, assume that $\sigma > 0$ and let v be a vector field along $f\colon M \to M'$ for which

$0 < \|v\| < \pi/(2\sqrt{\sigma})$ holds everywhere on M. By a simple calculation we have

$$\frac{1}{2}\langle \Psi(v,t) + \Psi(-v,t), v\rangle = \langle \partial f_*, v\rangle + \frac{1-\cos(\alpha t)}{\alpha^2}\operatorname{trace}\langle R'(f_*, v)\, dv, v\rangle = \\ = \langle \partial f_*, v\rangle - \|v\|(1-\cos(\alpha t))\left\langle f_*, d\left(\frac{v}{\|v\|}\right)\right\rangle .$$

Thus

$$\int_M \frac{1}{2\|v\|(1-\cos(\alpha t))}\langle \Psi(v,t) + \Psi(-v,t), v\rangle \operatorname{vol}(M) = \\ = \int_M \frac{\cos(\alpha t)}{1-\cos(\alpha t)}\left\langle \partial f_*, \frac{v}{\|v\|}\right\rangle \operatorname{vol}(M) .$$

Thus we obtain the following

PROPOSITION 3. – Let $f: M \to M'$ be a map, with $\sigma > 0$, and suppose that v is a vector field along f for which $0 < \|v\| < \pi/(2\sqrt{\sigma})$ holds.

(i) If $f_1 = \exp' \circ v$ and $f_{-1} = \exp \circ (-v)$ are harmonic then $\langle \partial f_*, v\rangle$ has a zero on M.

(ii) If $\langle \partial f_*, v\rangle$ is not identically zero and does not change its sign on M then at least one of the mappings f_1 and f_{-1} is nonharmonic.

EXAMPLE 2. – Let $f: S^1 \to S^2 \subset R^3$ be the canonical embedding onto the equator circle of S^2. If v is the vector field along f defined by a unit section of the normal bundle of f then $dv = 0$ and the mappings f_0 and $f_{\pi/2}$ are harmonic while f_t is nonharmonic for $0 < t < \pi/2$. This situation can be generalized as follows

THEOREM 5. – Let $f: M \to M'$ be a nonconstant harmonic map, with $\sigma > 0$, and suppose that v is a nonzero, parallel vector field along f. Then one of the following is valid:

(i) The map f_t is harmonic for all $t \geqq 0$ and f maps onto a closed geodesic γ of M' and v is tangent to γ.

(ii) The map f_t is nonharmonic for $0 < t < \pi/2\sqrt{\sigma}\|v\|$.

PROOF. – Because $dv = 0$ the map f_t os harmonic for some $t \geqq 0$ if and only if

$$\Psi(v,t) = \frac{\sin(\alpha t)}{\alpha}\operatorname{trace} R'(f_*, v) f_* + \frac{\sin(2\alpha t) - 2\sin(\alpha t)}{2\alpha\|v\|^2}\cdot \operatorname{trace}\langle R'(f_*, v) f_*, v\rangle v = 0 .$$

Suppose that f_{t_0} is harmonic for some $0 < t_0 < \pi/2\sqrt{\sigma}\|v\|$. Then

$$\langle \Psi(v, t_0), v\rangle = \frac{\sin(2\alpha t_0)}{2\alpha}\operatorname{trace}\langle R'(f_*, v) f_*, v\rangle = 0$$

and hence the bilinear form $\langle R'(f_*, v) f_*, v\rangle$ is zero. Thus rank $f \leqq 1$ and the remaining part of the proof follows from a result of J. H. Sampson, [7].

REMARK. - If $M \to M'$ is a map, with $\sigma > 0$, and v is a nonzero parallel vector field along f such that $\langle f_*(X_x), v_x\rangle = 0$ hold for every tangent vector $X_x \in T_x(M)$, $x \in M$, then

$$\Psi(v, t) = -\sigma \frac{\sin(2\alpha t)}{2\alpha} \|f_*\|^2 v ,$$

i.e. f is homotopic to a harmonic map.

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