SCIENCE CHINA
Mathematics

• ARTICLES • doi: 10.1007/s11425-000-0000-0

Dual mean Minkowski measures of symmetry for convex bodies

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Received August 21, 2015; accepted October 1, 2015

Abstract We introduce and study a sequence of geometric invariants for convex bodies in finite-dimensional spaces, which is in a sense dual to the sequence of mean Minkowski measures of symmetry proposed by the second author. It turns out that the sequence introduced in this paper shares many nice properties with the sequence of mean Minkowski measures, such as the sub-arithmeticity and the upper-additivity. More meaningfully, it is shown that this new sequence of geometric invariants, in contrast to the sequence of mean Minkowski measures which provides information on the shapes of lower dimensional sections of a convex body, provides information on the shapes of orthogonal projections of a convex body. The relations of these new invariants to the well-known Minkowski measure of asymmetry and their further applications are discussed as well.

Keywords geometric invariant, measure of symmetry, dual measure of symmetry, simplex, affine diameter

MSC(2010) 52A20, 46B99, 52A38, 52A39, 52A40

Citation: Guo Q, Toth G. Dual mean Minkowski measures of symmetry for convex bodies. Sci China Math, 2016, 59, doi: 10.1007/s11425-000-0000-0

1 Introduction

As one of the most important geometric invariants, the measure of symmetry (or asymmetry) of convex bodies (i.e., compact convex sets with nonempty interior in $\mathbb{R}^n$, the standard Euclidean space), formulated by Grünbaum in his well-known paper [3], has regained much attention in recent years (see [1, 2, 4–6, 9–11, 13, 14, 16] and references therein). So, some new measures have been found (see [1, 5, 11, 13, 16, 21]), and more properties of the known ones, including the stability and the relations with other kinds of geometric invariants, are revealed (see [1–3, 6, 14, 18, 19, 22]), and as consequences, some new geometric inequalities are established (see [1, 2, 4, 5, 9–11, 14, 22]).

In general, measures of symmetry (or asymmetry) can be used in geometry to measure how far a convex body (as a whole) is from some particular convex bodies, e.g., centrally symmetric convex bodies, convex cones or simplices. However, meaningfully, Toth [16] introduced a family of measures (functions) of symmetry σ_m (see below for definition), $m \geqslant 1$, called the mean Minkowski measures of symmetry, which, prior to most measures of symmetry (or asymmetry), measure not only convex bodies themselves but also their lower-dimensional sections. Roughly speaking, for a convex body K, its (m-th) mean Minkowski measure of symmetry σ_m is a function defined on int$K$, the interior of K, which, when $1 < m \leqslant n$, provides information on the shapes of m-dimensional sections of K.

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The properties and applications of the mean Minkowski measures of symmetry have been investigated in a series of papers (see [16–22]), where, as an application of the mean Minkowski measures, readers may find in particular a partial answer to the long-standing Grünbaum conjecture for the existence of $n+1$ affine diameters meeting at one point of a convex body (see [3]).

In this paper, we introduce another family of measures (functions) of symmetry σ_m°, $m \geqslant 1$, called the dual mean Minkowski measures of symmetry, which in a sense are dual to the mean Minkowski measures. It turns out that dual mean Minkowski measures share almost all nice properties with mean Minkowski measures and, in sharp contrast to the mean Minkowski measures, describe the shapes of orthogonal projections of a convex body. Furthermore, the dual mean Minkowski measures are relatively easier in computation than the mean Minkowski measures, and can also be applied to deal with the Grünbaum conjecture mentioned above as well (see [7]).

2 Notation and definition

Let $\mathcal{K}^n$ denote the family of all convex bodies in $\mathbb{R}^n$. For any subset $S \subset \mathbb{R}^n$, conv$S$ and cone$S$ denote the convex hull and the convex conical hull of S, respectively. lin$S$ denotes the linear subspace generated by S. A map $\boldsymbol{T}: \mathbb{R}^n \to \mathbb{R}^m$ is called affine if $\boldsymbol{T}(\lambda x + (1-\lambda)y) = \lambda \boldsymbol{T}(x) + (1-\lambda)\boldsymbol{T}(y)$ for any $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, in particular, an affine map $f: \mathbb{R}^n \to \mathbb{R}$ is called an affine function. It is known that $f: \mathbb{R}^n \to \mathbb{R}$ is affine if and only if $f(\cdot) = \langle u, \cdot\rangle + b$ for some unique $u \in \mathbb{R}^n$ and $b \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ denotes the classical inner product. Denote by $\mathrm{aff}(\mathbb{R}^n)$ the family of affine functions on $\mathbb{R}^n$ and by $\mathrm{Aff}(\mathbb{R}^n)$ the family of affine maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. We refer readers to [15] for other notation and terms.

The $(n-1)$-dimensional unit sphere is denoted by $\mathbb{S}^{n-1}$. An n-dimensional simplex (n-simplex for brevity) is denoted by Δ_n, i.e., $\Delta_n := \mathrm{conv}\{v_0, v_1, \ldots, v_n\}$, where $v_0, v_1, \ldots, v_n \in \mathbb{R}^n$, called the vertices of Δ_n, are affinely independent.

We recall the well-known Minkowski measure of asymmetry: Given a convex body $K \in \mathcal{K}^n$ and $x \in \mathrm{int}K$, for a hyperplane H through x and the pair of support hyperplanes H_1, H_2 of C parallel to H, let $r(H, x)$ be the ratio, not less than 1, in which H divides the distance between H_1 and H_2. Note

$$\mathrm{as}_\infty(x) = \mathrm{as}_\infty(K, x) := \max\{r(H, x) : H \ni x\},$$

and the Minkowski measure $\mathrm{as}_\infty(K)$ of asymmetry of K is defined by (see [3, 12])

$$\mathrm{as}_\infty(K) = \min_{x \in \mathrm{int}(K)} \mathrm{as}_\infty(K, x).$$

A point $x \in \mathrm{int}(K)$ such that $\mathrm{as}_\infty(K, x) = \mathrm{as}_\infty(K)$ is called a Minkowski (or ∞-) critical point (of K). The set of all ∞-critical points of K is denoted by $\mathcal{C}_\infty$. Another equivalent definition is as follows: Let $l := pq$ be a chord of K passing through x, where $p, q \in \boldsymbol{bd}K$, the boundary of K. If defining $\gamma(K, x) := \max_{l \ni x} \frac{d(p,x)}{d(x,q)}$, where $d(\cdot, \cdot)$ is the Euclidean metric, then $\gamma(K, x) = \mathrm{as}_\infty(K, x)$ (see [12]) and so $\mathrm{as}_\infty(K) = \min_{x \in \mathrm{int}(K)} \gamma(K, x)$.

It is known that for any $K \in \mathcal{K}^n$, $1 \leqslant \mathrm{as}_\infty(K) \leqslant n$, and $\mathrm{as}_\infty(K) = 1$ iff K is (centrally) symmetric and $\mathrm{as}_\infty(K) = n$ iff K is an n-dimensional simplex (see [3, 12]).

Next, we recall the definition of the mean Minkowski measure (function) of symmetry introduced in [16]. Some notation is needed first.

Let $K \in \mathcal{K}^n$ and $x \in \mathrm{int}K$. A multi-set $\{c_0, c_1, \ldots, c_m\}$ ($m \geqslant 1$ and repetitions are allowed), where $c_0, c_1, \ldots, c_m \in \boldsymbol{bd}K$, is called an *m-configuration* of K with respect to (w.r.t. for brevity) x if $x \in \mathrm{conv}\{c_0, c_1, \ldots, c_m\}$. Denote by $\mathcal{C}_m(x) = \mathcal{C}_{K,m}(x)$ the family of m-configurations of K w.r.t. x.

Definition 2.1 (See [16]). Given $K \in \mathcal{K}^n$, for each $m \geqslant 1$, we define its (m-th) mean measure (function) of symmetry $\sigma_m = \sigma_{K,m} : \mathrm{int}K \to \mathbb{R}$ by

$$\sigma_m(x) := \inf_{\{c_0, \ldots, c_m\} \in \mathcal{C}_m(x)} \sum_{i=0}^{m} \frac{1}{\Lambda(c_i, x) + 1}, \quad x \in \mathrm{int}K,$$

where $\Lambda(c,x)=\Lambda_K(c,x):=\frac{d(c,x)}{d(c^o,x)}$ is the distortion and $c^o\in \boldsymbol{bd}K$ denotes the opposite point of $c\in\boldsymbol{bd}K$ against x.

Clearly, for each $m\geqslant 1$, σ_m is affinely invariant. The following theorem was proved in [17].

Theorem A (See [17]). *Let $K\in\mathcal{K}^n$. For $m\geqslant 1$, we have*

$$1\leqslant\sigma_m\leqslant\frac{m+1}{2}.$$

If $m\geqslant 2$, then $\sigma_m(x)=(m+2)/2$ for some $x\in\mathrm{int}K$ iff K is symmetric with respect to x. If $\sigma_m(x)=1$ for some $x\in\mathrm{int}K$, then $m\leqslant n$ and K has an m-dimensional simplicial intersection across x, i.e., there is an m-dimensional hyperplane H such that $x\in H$ and $K\cap H$ is an m-simplex. Conversely, if K has a simplicial intersection with an m-dimensional hyperplane H, then $\sigma_m=1$ identically on $K\cap H$.

From Theorem A, we see that the mean Minkowski measures provide indeed information about the lower dimensional sections of a convex body.

Now we introduce a dual measure to the mean Minkowski measure, called the dual mean Minkowski measure. In order to do so, we need some more notation. Given $K\in\mathcal{K}^n$, we define the set $K^a_{[0,1]}$ by

$$K^a_{[0,1]}:=\{f\in\mathrm{aff}(\mathbb{R}^n)\mid f(K)=[0,1]\}.$$

It is easy to see that if $f\in K^a_{[0,1]}$, then $\{f=0\}$ and $\{f=1\}$ are a pair of (parallel) support hyperplanes of K, from which it follows that $\mathrm{as}_\infty(K,x)=\max\{\frac{1-f(x)}{f(x)}\mid f\in K^a_{[0,1]}\}, x\in\mathrm{int}K$ (see [4,5]).

Given $K\in\mathcal{K}^n$, for each $m\geqslant 1$, we define its *m-support configuration* in the following way: A multi-set $\{f_0,\dots,f_m\}\subset K^a_{[0,1]}$ (repetitions are allowed) is called an m-support configuration of K if $\bigcap_{i=0}^m\{f_i\leqslant 0\}=\emptyset$, where $\{f_i\leqslant 0\}:=\{x\in\mathbb{R}^n\mid f_i(x)\leqslant 0\}$. The family of m-support configurations of K is denoted by $\mathcal{C}^\circ_m=\mathcal{C}^\circ_{K,m}$.

Remark 2.2. In contrast to $\mathcal{C}_m(x)$, $\mathcal{C}^\circ_m$ does not depend on any point in the interior of K.

Definition 2.3. Let $K\in\mathcal{K}^n$. For each $m\geqslant 1$, its (m-th) dual mean Minkowski measure (function) $\sigma^\circ_m=\sigma^\circ_{K,m}:\mathrm{int}K\to\mathbb{R}$ is defined by

$$\sigma^\circ_m(x):=\inf\left\{\sum_{i=0}^m f_i(x)\mid\{f_0,\dots,f_m\}\in\mathcal{C}^\circ_m\right\},\quad x\in\mathrm{int}K.$$

A point $x^*\in\mathrm{int}K$ satisfying $\sigma^\circ_m(x^*)=\sup_{x\in\mathrm{int}K}\sigma^\circ_m(x)$ is called a σ°_m-critical point of K.

Remark 2.4. 1° The dual mean Minkowski measure is indeed a dual concept of the mean Minkowski measure (see Corollary 3.2).

2° Each σ°_m is concave in $\mathrm{int}K$ since it is the infimum of some concave functions $\sum_{i=0}^m f_i(x)$, whereas σ_m is not concave in general (see [19]). Thus σ°_m and σ_m do not coincide in general.

3° Since σ°_m is concave, $\sigma^\circ_m(x)\geqslant 1$, $x\in\mathrm{int}K$ (see Corollary 3.3) and $\lim_{x\to\boldsymbol{bd}K}\sigma^\circ_m(x)=1$ (see Proposition 4.2), there exists at least one σ°_m-critical point.

4° $\sigma^\circ_1\equiv 1$ trivially in $\mathrm{int}K$ since $\{f_0,f_1\}\in\mathcal{C}^\circ_1$ iff $f_1=1-f_0$.

Among other conclusions, one of the main results in this paper is the following theorem.

Theorem B. *Let $K\in\mathcal{K}^n$. For $m\geqslant 1$, we have $1\leqslant\sigma^\circ_m\leqslant\frac{m+1}{2}$. If $m\geqslant 2$, then $\sigma^\circ_m(x)=\frac{(m+1)}{2}$ for some $x\in\mathrm{int}K$ iff K is a symmetric body centered at x. If $\sigma^\circ_m(x)=1$ for some $x\in\mathrm{int}K$, then $m\leqslant n$, $\sigma^\circ_m\equiv 1$ and K has an m-dimensional simplicial projection, i.e., there is a projection $P_H:\mathbb{R}^n\to H$, where H is an m-dimensional subspace, such that $P_H(K)$ is an m-simplex. Conversely, if K has an m-dimensional simplicial projection* ($2\leqslant m\leqslant n$), *then $\sigma_m\equiv 1$.*

From Theorem B, we see that the dual mean Minkowski measures provide indeed information about the lower-dimensional orthogonal projections of a convex body.

The paper is organized as following: Section 3 discusses the characteristics and properties of the support configurations. Section 4 studies the basic properties, such as sub-arithmeticity and upper-additivity etc, of the dual mean Minkowski measure sequences. Finally, Section 5 is devoted to the proof of Theorem B.

3 Properties of support configurations

In this section, we discuss the properties of support configurations and show some of their characteristics. We prove first the following theorem.

Theorem 3.1. *Let $f_i(x) \in K^a_{[0,1]}, i = 0, 1, \dots, m$, where $f_i(\cdot) = \langle u_i, \cdot\rangle + b_i$. Then, the following statements are equivalent*:

(1) $\{f_i\}_{i=0}^m \in \mathcal{C}_m^\circ$.

(2) *For each $u \in \mathbb{R}^n$, $\langle u_i, u\rangle \leqslant 0$ for some i.*

(3) $o \in \mathrm{ri}(\mathrm{conv}\{u_{i_0}, \dots, u_{i_l}\})$ *for some affinely independent $u_{i_0}, \dots, u_{i_l}$, $1 \leqslant l \leqslant \min\{m, n\}$, i.e., there are positive α_{i_k} such that $\sum_{k=0}^l \alpha_{i_k} u_{i_k} = o$.*

(4) $\mathrm{cone}\{u_{i_0}, u_{i_1}, \dots, u_{i_l}\} = \mathrm{lin}\{u_{i_0}, u_{i_1}, \dots, u_{i_l}\}$ *for some affinely independent $u_{i_0}, u_{i_1}, \dots, u_{i_s}$, $1 \leqslant l \leqslant \min\{m, n\}$.*

In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let $f_i(x) \in K^a_{[0,1]}, i = 0, 1, \dots, m$, where $f_i(\cdot) = \langle u_i, \cdot\rangle + b_i$. Then, for any $(u_{m+1}, b_{m+1}) \in \mathrm{cone}\{(u_0, b_0), \dots, (u_m, b_m)\}$, we have*

$$\bigcap_{i=0}^m \{f_i \leqslant 0\} = \bigcap_{i=0}^{m+1} \{f_i \leqslant 0\},$$

where $f_{m+1}(\cdot) = \langle u_{m+1}, \cdot\rangle + b_{m+1}$.

Proof. If $\bigcap_{i=0}^m \{f_i \leqslant 0\} = \emptyset$, then the equality is obvious. If $\bigcap_{i=0}^m \{f_i \leqslant 0\} \neq \emptyset$, then this is just a reformulation of the well-known generalized Farkas lemma (see [8, p. 60]). □

Proof of Theorem 3.1. (1) ⇒ (2) Let $\{f_i\}_{i=0}^m \in \mathcal{C}_m^\circ$ and $u \in \mathbb{R}^n$. If $\langle u_i, u\rangle > 0$ or $\langle u_i, -u\rangle < 0$ for all i, then $f_i(-\lambda u) = \langle u_i, -\lambda u\rangle + b_i \to -\infty$ as $\lambda \to +\infty$ for all i. Thus, $-\lambda u \in \bigcap_{i=0}^m \{f_i \leqslant 0\}$ for sufficiently large λ, a contradiction to (1)!

(2) ⇒ (3) Consider the linear subspace $V := \mathrm{cone}\{u_i\}_{i=0}^m \cap (-\mathrm{cone}\{u_i\}_{i=0}^m)$. If $V = \{o\}$, then by the separation theorem for cones there is $u \in \mathbb{R}^n$ such that $\{u_i\}_{i=0}^m \in \{x \mid \langle u, x\rangle > 0\}$, which contradicts (2). So $\dim V \geqslant 1$. Now, it is easy to see by the definition that $V = \mathrm{cone}\{u_{i_0}, \dots, u_{i_s}\} = \mathrm{lin}\{u_0, \dots, u_s\}$ for some $u_{i_0}, \dots, u_{i_s}$ $(1 \leqslant s \leqslant m)$.

Thus, since $-u_{i_0} \in \mathrm{lin}\{u_{i_k}\}_{k=0}^s = \mathrm{cone}\{u_{i_k}\}_{k=0}^s$, we have $-u_{i_0} = \alpha_0' u_{i_0} + \sum_{k=1}^s \alpha_{i_k} u_{i_k}$ with $\alpha_0', \alpha_{i_k} \geqslant 0$. Thus $o = \sum_{k=0}^s \alpha_{i_k} u_{i_k}$, where $\alpha_{i_0} := 1 + \alpha_0' > 0$. Now, let l be the smallest positive integer such that $o = \sum_{k=0}^l \alpha_{i_k} u_{i_k}$ for some $u_{i_0}, \dots, u_{i_l}$ and with $\alpha_{i_k} > 0$. Clearly $l \geqslant 1$. We claim that $u_{i_0}, \dots, u_{i_l}$ are affinely independent. Suppose $u_{i_0}, \dots, u_{i_l}$ are not affinely dependent, then $\sum_{k=0}^l \beta_{i_k} u_{i_k} = o$ for some (not all zero) $\beta_{i_0}, \dots, \beta_{i_l}$ with $\sum_{k=0}^l \beta_{i_k} = 0$. Let

$$\lambda := \min\left\{ \frac{-\alpha_{i_k}}{\beta_{i_k}} \,\middle|\, \beta_{i_k} < 0 \right\} = (\mathrm{say}) \frac{-\alpha_{i_l}}{\beta_{i_l}}$$

then

$$o = \sum_{k=0}^l \alpha_{i_k} u_{i_k} + \sum_{k=0}^l \lambda \beta_{i_k} u_{i_k} = \sum_{k=0}^l (\alpha_{i_k} + \lambda\beta_{i_k}) u_{i_k} = \sum_{k=0}^{l-1} (\alpha_{i_k} + \lambda\beta_{i_k}) u_{i_k},$$

where $\alpha_{i_k} + \lambda\beta_{i_k} \geqslant 0$ $(0 \leqslant k \leqslant l-1)$ and at least one of them is positive, a contradiction to the choice of l.

Thus,

$$o = \frac{1}{\sum_{k=0}^l \alpha_{i_k}} \cdot \sum_{k=0}^l \alpha_{i_k} u_{i_k} = \sum_{k=0}^l \frac{\alpha_{i_k}}{\sum_{k=0}^l \alpha_{i_k}} u_{i_k} \in \mathrm{ri}(\mathrm{conv}\{u_{i_0}, \dots, u_{i_l}\}),$$

where $l \leqslant \min\{m, n\}$ clearly.

(3) ⇒ (4) Without loss of generality, suppose that $u_0, \dots, u_l$, $1 \leqslant l \leqslant \min\{m, n\}$, are affinely independent and $o \in \mathrm{ri}(\mathrm{conv}\{u_0, \dots, u_l\})$, i.e., $o = \sum_{i=0}^l \alpha_i u_i$ with $\alpha_i > 0$. Then for each $0 \leqslant i \leqslant l$, we have $-u_i = \sum_{j\neq i}^l \frac{\alpha_j}{\alpha_i} u_j \in \mathrm{cone}\{u_0, \dots, u_l\}$. Thus $\mathrm{lin}\{u_0, \dots, u_l\} = \mathrm{cone}\{u_0, \dots, u_l\}$.

(4) $\Rightarrow$ (1) Without loss of generality, suppose $\mathrm{cone}\{u_0,\dots,u_l\} = \mathrm{lin}\{u_0,\dots,u_l\}$ with affinely independent $u_0,\dots,u_l$ for some $1 \leqslant l \leqslant \min\{m,n\}$, then $-u_0 = \sum_{i=0}^{l} \alpha_i u_i$, where $\alpha_i \geqslant 0$ with at least one α_i positive. Thus, setting $f_{m+1}(\cdot) := \langle -u_0, \cdot\rangle + \sum_{i=0}^{l} \alpha_i b_i$, we have, by Lemma 3.1,

$$\bigcap_{i=0}^{m}\{f_i \leqslant 0\} = \bigcap_{i=0}^{m+1}\{f_i \leqslant 0\} = \emptyset$$

since $\{f_0 \leqslant 0\} \cap \{f_{m+1} \leqslant 0\} = \emptyset$ (observing that

$$\begin{aligned}\inf_{x\in \mathrm{int}K} f_{m+1}(x) &= \inf_{x\in \mathrm{int}K} \sum_{i=0}^{t} \alpha_i \langle u_i, x\rangle + \sum_{i=0}^{t} \alpha_i b_i \\ &\geqslant \sum_{i=0}^{t} \alpha_i \inf_{x\in \mathrm{int}K}\langle u_i, x\rangle + \sum_{i=0}^{t} \alpha_i b_i \\ &= \sum_{i=0}^{t} \alpha_i \Big(\inf_{x\in \mathrm{int}K}\langle u_i, x\rangle + b_i\Big) = 0\end{aligned}$$

and so $\{f_{m+1} \leqslant 0\} \subset \{f_0 \geqslant 1\}$). Therefore $\{f_0, f_1,\dots,f_m\} \in \mathcal{C}_m^\circ$. □

We now present some other basic properties of support configurations.

Proposition 3.3. *Let $K \in \mathcal{K}^n$, $f_0,\dots,f_m \in K^a_{[0,1]}, m \geqslant 1$.*

(1) $\{f_0,\dots,f_m\} \in \mathcal{C}_m^\circ$ *iff* $\{1-f_0,\dots,1-f_m\} \in \mathcal{C}_m^\circ$.

(2) $\bigcap_{i=0}^m \{f_i \leqslant 0\} = \emptyset$ *iff* $\bigcap_{i=0}^m\{f_i < 0\} = \emptyset$.

(3) $\mathcal{C}_m^\circ$ *is compact in* $\mathbb{R}^{(n+1)(m+1)}$.

Proof. (1) Write $f_i(\cdot) = \langle u_i, \cdot\rangle + b_i$, then $1 - f_i(x) = \langle -u_i, x\rangle - b_i + 1, i = 0, 1,\dots,m$. Thus $\{f_0,\dots,f_m\} \in \mathcal{C}_m^\circ$, by Theorem 3.1(2), iff for each $-u \in \mathbb{R}^n$, $\langle -u, u_i\rangle \leqslant 0$ for some $0 \leqslant i \leqslant m$, i.e., iff for each $u \in \mathbb{R}^n$, $\langle u, -u_i\rangle \leqslant 0$ for some $0 \leqslant i \leqslant m$, and so, by Theorem 3.1 again, iff $\{1 - f_0,\dots,1 - f_m\} \in \mathcal{C}_m^\circ$.

(2) $\bigcap_{i=0}^m\{f_i \leqslant 0\} = \emptyset$ implies clearly $\bigcap_{i=0}^m\{f_i < 0\} = \emptyset$.

If $\bigcap_{i=0}^m\{f_i \leqslant 0\} \neq \emptyset$, choosing $x_1 \in \bigcap_{i=0}^m\{f_i \leqslant 0\}$ and $x_2 \in \mathrm{int}K$ (so $f_i(x_2) > 0$ for all i), we have, since $f_i(\lambda x_1 + (1-\lambda)x_2) = \lambda f_i(x_1) + (1-\lambda)f_i(x_2) \to -\infty$ as $\lambda \to +\infty$ for each i, $f_i(\lambda_0 x_1 + (1-\lambda_0)x_2) < 0$ for some $\lambda_0 > 0$ and all i. Thus, $\lambda_0 x_1 + (1-\lambda_0)x_2 \in \bigcap_{i=0}^m\{f_i < 0\}$, i.e., $\bigcap_{i=0}^m\{f_i < 0\} \neq \emptyset$.

(3) Since $K^a_{[0,1]}$ is compact in $\mathbb{R}^{n+1}$ (see [9, Lemma 1]) and so is $K^a_{[0,1]} \times \cdots \times K^a_{[0,1]}$ ($(m+1)$-fold), we need only to show that $\mathcal{C}_m^\circ \in K^a_{[0,1]} \times \cdots \times K^a_{[0,1]}$ is closed.

Let $\{f_0^{(k)},\dots,f_m^{(k)}\} \in \mathcal{C}_m^\circ, k = 1, 2,\dots,$ and $\{f_0^{(k)},\dots,f_m^{(k)}\} \to \{f_0,\dots,f_m\} \in K^a_{[0,1]} \times \cdots \times K^a_{[0,1]}$ as $k \to \infty$, which is equivalent to $f_i^{(k)} \to f_i$ for each i. Now, suppose $\{f_0,\dots,f_m\} \notin \mathcal{C}_m^\circ$, i.e., $\bigcap_{i=0}^m\{x \mid f_i(x) \leqslant 0\} \neq \emptyset$, then by (2) just proved above, there exists $x_0 \in \bigcap_{i=0}^m\{x \mid f_i(x) < 0\}$, i.e., $f_i(x_0) < 0$ for each i. Thus, since $f_i^{(k)}(x_0) \to f_i(x_0)$ as $k \to \infty$, we conclude that there is k_0 such that $f_i^{(k)}(x_0) \leqslant 0$ for all $k \geqslant k_0$ and all i, which contradicts that $\{f_0^{(k)},\dots,f_m^{(k)}\} \in \mathcal{C}_m^\circ$, $k \geqslant k_0$. □

Corollary 3.4. *The infimum in the definition of σ_m° is attainable, i.e., for given $x \in \mathrm{int}K$, there is an m-support configuration $\{f_0,\dots,f_m\} \in \mathcal{C}_{K,m}^\circ$ such that $\sum_{i=0}^m f_i(x) = \sigma_m^\circ(x)$.*

Proof. This follows from Proposition 3.1(3) and the fact that $\sum_{i=0}^m f_i(x)$ is continuous w.r.t. $\{f_0,\dots,f_m\}$. □

An m-support configuration $\{f_i\}$ such that $\sum_{i=0}^m f_i(x) = \sigma_m^\circ(x)$ is called an *m-minimizer* w.r.t. $x \in \mathrm{int}K$.

The next corollary shows a kind of duality between σ_m and σ_m°. Before stating the corollary, we need some preparations.

Given $K \in \mathcal{K}^n$ and $x \in \mathrm{int}K$, we define the support function $h_x(\cdot) = h_x(K,\cdot)$ based at x of K by

$$h_x(K, u) := \sup\{\langle u, y - x\rangle \mid y \in K\}, \quad u \in \mathbb{R}^n,$$

the gauge function $g_x(\cdot) = g_x(K, \cdot)$ based at x of K by

$$g_x(K,u) := \inf\{\lambda \geqslant 0 \mid u \in \lambda(K-x)\}, \quad u \in \mathbb{R}^n,$$

and the dual body K^x based at x of K by

$$K^x := \{y \mid \langle y, z-x\rangle \leqslant 1, z \in K\} + x.$$

Clearly, when $x = o \in \mathrm{int}K$, h_o, g_o and K^o are exactly the classical ones, and

$$\begin{aligned} &h_x(K,u) = h_o(K-x,u), \quad g_x(K,u) = g_o(K-x,u), \quad K^x := (K-x)^o + x,\\ &h_x(K^x,u) = h_o((K-x)^o,u), \quad g_x(K^x,u) = g_o((K-x)^o,u). \end{aligned}$$

[15, Lemma 1.7.13] states an elegant relation between the support and the gauge function: If $o \in \mathrm{int}K$, then for any $u \in \mathbb{R}^n$, $g_o(K,u) = h_o(K^o,u)$.

Corollary 3.5. *Let $K \in \mathcal{K}^n$ and $m \geqslant 1$. Then,*

$$\sigma_{K,m}(x) = \sigma^\circ_{K^x,m}(x), \quad \sigma^\circ_{K,m}(x) = \sigma_{K^x,m}(x).$$

Proof. We point out first a fact that for each $u \in \mathbb{R}^n \setminus \{o\}$, there are (unique) $\mu > 0$ and $b \in \mathbb{R}$ such that $f(\cdot) := \langle \mu u, \cdot\rangle + b \in K^a_{[0,1]}$ (choosing $b = b_1/a$ and $\mu = 1/a$, where $b_1 := -\inf_{x\in K}\langle u, x\rangle$ and $a := \sup_{x\in K}\langle u,x\rangle + b_1$).

Then, for $c_i \in \boldsymbol{bd}K$, defining $f_i(\cdot) := \langle \mu_i(c_i - x), \cdot\rangle + b_i \in (K^x)^a_{[0,1]}$, $0 \leqslant i \leqslant m$, where $\mu_i > 0$ as mentioned above, we have

$$\begin{aligned} &\{c_0, c_1, \dots, c_m\} \in \mathcal{C}_{K,m}(x)\\ &\quad \Leftrightarrow x \in \mathrm{conv}\{c_0, c_1, \dots, c_m\} \text{ (by definition)}\\ &\quad \Leftrightarrow x = \sum_{k=0}^{l} \alpha_{i_k} c_{i_k}, \text{ where } l \geqslant 1, \{c_{i_k}\} \subset \{c_i\}, \alpha_{i_k} > 0, \sum_{k=0}^{l} \alpha_{i_k} = 1, \text{ (by } x \notin \boldsymbol{bd}K)\\ &\quad \Leftrightarrow o = \sum_{k=0}^{l} \frac{\alpha_{i_k}}{\mu_{i_k}} \mu_{i_k}(c_{i_k} - x) \Leftrightarrow \{f_0, f_1, \dots, f_m\} \in \mathcal{C}^\circ_{K^x,m} \text{ (by Theorem 3.1).} \end{aligned}$$

Next, observing that for $c \in \boldsymbol{bd}K$, $\frac{d(x,c_i)}{d(x,c_i^\circ)} = g_o(K-x, x-c)$ and $g_o(K-x, c-x) = 1$ since $c - x, c^\circ - x \in \boldsymbol{bd}(K-x)$, we obtain

$$\begin{aligned} \frac{1}{\Lambda(c_i,x)+1} &= \frac{1}{\frac{d(x,c_i)}{d(x,c_i^\circ)}+1} = \frac{g_o(K-x, c_i-x)}{g_o(K-x,x-c_i) + g_o(K-x,c_i-x)}\\ &= \frac{h_o((K-x)^\circ, c_i - x)}{h_o((K-x)^\circ, x-c_i) + h_o((K-x)^\circ, c_i-x)} \text{ (by [15, Lemma 1.7.13])}\\ &= \frac{h_x(K^x, c_i-x)}{h_x(K^x, x-c_i) + h_x(K^x, c_i - x)} = 1 - f_i(x). \end{aligned}$$

Hence, with the help of Proposition 3.1, we have

$$\begin{aligned} \sigma_{K,m}(x) &= \inf_{\{c_0,c_1,\dots,c_m\}\in\mathcal{C}_{K,m}(x)} \sum_{i=0}^{m} \frac{1}{\Lambda(c_i,x)+1}\\ &= \inf_{\{f_0,f_1,\dots,f_m\}\in\mathcal{C}^\circ_{K^x,m}} \sum_{i=0}^{m} (1 - f_i(x)) = \inf_{\{f_0,f_1,\dots,f_m\}\in\mathcal{C}^\circ_{K^x,m}} \sum_{i=0}^{m} f_i(x) = \sigma^\circ_{K^x,m}(x). \end{aligned}$$

The second equality follows from the fact that $(K^x)^x = K$. □

One kind of particular support configurations will play an important role in the study of dual mean Minkowski measures of symmetry.

Definition 3.6. Let $1 \leqslant m \leqslant n$. For $1 \leqslant m \leqslant n$, an $\{f_0, \ldots, f_m\} \in \mathcal{C}_m^\circ$, where $f_i(\cdot) = \langle u_i, \cdot\rangle + b_i$, is called simplicial if $\bigcap_{i=0}^m \{f_i \geqslant 0\} \cap \operatorname{lin}\{u_i\}_{i=0}^m$ is an m-simplex, where an 1-simplex means a segment.

Theorem 3.7. *Let $\{f_0, \ldots, f_m\} \in \mathcal{C}_m^\circ$ with $f_i(\cdot) = \langle u_i, \cdot\rangle + b_i$ $(1 \leqslant m \leqslant n)$. Then the following are equivalent*:

(1) *$\{f_0, \ldots, f_m\}$ is simplicial.*

(2) *$u_0, \ldots, u_m$ are affinely independent and* $\operatorname{cone}\{u_i\}_{i=0}^m = \operatorname{lin}\{u_i\}_{i=0}^m$.

(3) *$\sum_{i=0}^m \alpha_i u_i = o$ for some positive $\alpha_0, \ldots, \alpha_m$ and if $\sum_{i=1}^m \beta_i u_i = o$ with non-negative β_i, then either $\beta_i > 0$ for all i or $\beta_i = 0$ for all i.*

(4) *$\{f_0, \ldots, f_m\}$ has no proper sub-support configurations.*

Proof. (1) $\Rightarrow$ (2) If $\{f_0, \ldots, f_m\}$ is simplicial, then $u_0, \ldots, u_m$ are exactly the $m+1$ inner normals of facets of the m-simplex $\Delta_m := \bigcap_{i=0}^m \{f_i \geqslant 0\} \cap \operatorname{lin}\{u_i\}_{i=0}^m$. So $\{u_i\}_{i=0}^m$ are affinely independent and $\dim(\operatorname{lin}\{u_i\}_{i=0}^m) = m$. Suppose that $\operatorname{cone}\{u_i\}_{i=0}^m \subsetneq \operatorname{lin}\{u_i\}_{i=0}^m$. Then by the Separation Theorem for cones, $\operatorname{cone}\{u_i\}_{i=0}^m \subset \{x \in \operatorname{lin}\{u_i\}_{i=0}^m \mid \langle u, x\rangle \geqslant 0\}$ for some $u \in \operatorname{lin}\{u_i\}_{i=0}^m$. Thus, choosing $x_0 \in \Delta_m$, we have

$$f_i(x_0 + tu) = \langle u_i, x_0 + tu\rangle + b_i = \langle u_i, x_0\rangle + b_i + t\langle u_i, u\rangle \geqslant 0$$

for all $t \geqslant 0$ and all i, i.e., $\{x_0 + tu \mid t \geqslant 0\} \subset \Delta_m$, which contradicts the boundedness of Δ_m in $\operatorname{lin}\{u_i\}_{i=0}^m$ (noticing that $x_0 + tu \in \operatorname{lin}\{u_i\}_{i=0}^m$ for all t).

(2) $\Rightarrow$ (3) Since $\operatorname{cone}\{u_i\}_{i=0}^m = \operatorname{lin}\{u_i\}_{i=0}^m$, we have $-u_0 = \sum_{i=0}^m \gamma_i u_i$ for some $\gamma_i \geqslant 0$. Thus $\sum_{i=0}^m \alpha_i u_i = o$, where $\alpha_0 = 1 + \gamma_0 > 0$ and $\alpha_i = \gamma_i \geqslant 0$ for $1 \leqslant i \leqslant m$.

Before showing that all α_i above are positive, we show first the second conclusion. Assume that $\sum_{i=0}^m \beta_i u_i = o$ for some non-negative $\beta_0, \ldots, \beta_m$. Suppose that some β_i are zeros and some β_i are positive. Without loss of generality, we assume $\beta_m = 0$ (and so $\sum_{i=0}^{m-1} \beta_i \neq o$). Observing that $-u_m = \sum_{i=0}^m \beta_i' u_i$ or equivalently $\sum_{i=0}^{m-1} \beta_i' u_i + (1 + \beta_m')u_m = o$ for some $\beta_i' \geqslant 0$, we have

$$\sum_{i=0}^{m-1} (\mu\beta_i - \beta_i')u_i - (1 + \beta_m')u_m = o \quad \text{and} \quad \sum_{i=0}^{m-1} (\mu\beta_i - \beta_i') - (1 + \beta_m') = 0,$$

where $\mu = (\sum_{i=0}^{m-1} \beta_i)^{-1}(\sum_{i=0}^{m-1} \beta_i' + (1 + \beta_m'))$, which contradicts the affine independence of $u_0, \ldots, u_m$ since $1 + \beta_m' \neq 0$.

Now the fact that all α_i are positive is just a simple consequence of the second conclusion.

(3) $\Rightarrow$ (4) Without loss of generality, suppose $\{f_0, \ldots, f_{m_1}\} \in \mathcal{C}_{m_1}^\circ$, where $1 \leqslant m_1 < m$, then by (3) in Theorem 3.1, there are affinely independent $u_{i_0}, \ldots, u_{i_l}$, $1 \leqslant l \leqslant \min\{m_1, n\} < m$, such that $o \in \operatorname{ri}(\operatorname{conv}\{u_{i_k}\}_{k=0}^l)$, i.e., $\sum_{k=1}^l \alpha_{i_k} u_{i_k} = o$ with $\alpha_{i_k} > 0$, which contradicts (3) since $l < m$.

(4) $\Rightarrow$ (1) By Theorem 3.1, $o \in \operatorname{ri}(\operatorname{conv}\{u_{i_0}, \ldots, u_{i_l}\})$ for some affinely independent $u_{i_0}, \ldots, u_{i_l}$, $1 \leqslant l \leqslant m$, which in turn shows $\{f_{i_0}, \ldots, f_{i_l}\} \in \mathcal{C}_l^\circ$ by Theorem 3.1 again. Thus $l = m$ by (4) and so $o = \sum_{k=0}^m \alpha_i u_i$ with all $\alpha_i > 0$. Now we claim that the non-empty set $\Delta_m := \bigcap_{i=0}^m \{f_i \geqslant 0\} \cap \operatorname{lin}\{u_i\}_{i=0}^m$ is bounded in the m-dimensional subspace $\operatorname{lin}\{u_i\}_{i=0}^m$ which means that $\{f_0, \ldots, f_m\}$ is simplicial.

Suppose that Δ_m is not bounded in $\operatorname{lin}\{u_i\}_{i=0}^m$, then $\{x_0 + tu \mid t \geqslant 0\} \subset \Delta_m$ for some $x_0 \in \Delta_m$ and non-zero $u \in \operatorname{lin}\{u_i\}_{i=0}^m$ (so $f_i(x_0 + tu) \geqslant 0$ for all $t > 0$ and all i). However, since $0 = \sum_{i=0}^m \alpha_i \langle u_i, u\rangle$, we have $\langle u_{i_0}, u\rangle < 0$ for some i_0 and further

$$0 \leqslant f_{i_0}(x_0 + tu) = \langle u_{i_0}, x_0\rangle + t\langle u_{i_0}, u\rangle + b_{i_0} \to -\infty \quad \text{as} \quad t \to +\infty$$

a contradiction. □

The next proposition shows that the simplicial support configurations are not very special.

Proposition 3.8. *Any $\{f_0, \ldots, f_m\} \in \mathcal{C}_m^\circ$ $(m \geqslant 1)$ has a simplicial sub-support configuration $\{f_{i_0}, \ldots, f_{i_l}\} \in \mathcal{C}_l^\circ$, where $1 \leqslant l \leqslant \min\{m, n\}$.*

Proof. Denote $f_i(\cdot) = \langle u_i, \cdot\rangle + b_i$. Then, let l be the smallest positive integer such that $\sum_{k=0}^l \alpha_{i_k} u_{i_k} = o$ for some $\{u_{i_k}\}_{k=0}^l \subset \{u_i\}_{i=0}^m$ and $\alpha_{i_k} > 0$, $0 \leqslant k \leqslant l$ (by Theorem 3.1, such l exists and $1 \leqslant l \leqslant \min\{m, n\}$). Thus, $\{f_{i_0}, \ldots, f_{i_l}\} \in \mathcal{C}_l^\circ$ by Theorem 3.1 again, and further $\{f_{i_0}, \ldots, f_{i_l}\}$ is l-simplicial by Theorem 3.2(3). □

We end this section with the following proposition and its corollary.

Proposition 3.9. *Let $K \in \mathcal{K}^n$. If $\{f_0,\dots,f_n\} \in \mathcal{C}^\circ_{K,n}$ is n-simplicial, then $\sum_{i=0}^n f_i(x) \geqslant 1$ for all $x \in \mathrm{int}K$ and equality holds for some $x \in \mathrm{int}K$ iff $K = \bigcap_{i=0}^n \{x \mid f_i(x) \geqslant 0\}$, i.e., K is an n-simplex.*

In order to prove Proposition 3.3, we need the following well-known fact and we repeat the proof here for completeness (see [4]).

Lemma 3.10. *Let $\Delta \in \mathcal{K}^n$ be an n-simplex with vertices $v_0,\dots,v_n$ and $g_i \in \Delta^a_{[0,1]}$ be such that $g_i(v_j) = \delta_{ij}$, the Kronecker symbol. Then $\sum_{i=0}^n g_i \equiv 1$ in $\mathbb{R}^n$.*

Proof. Observing that for any $x \in \mathbb{R}^n$, $x = \sum_{i=0}^n \alpha_i v_i$ for some $\alpha_0,\dots,\alpha_n \in \mathbb{R}$ with $\sum_{i=0}^n \alpha_i = 1$, we have,

$$\sum_{i=0}^n g_i(x) = \sum_{i=0}^n g_i\left(\sum_{j=0}^n \alpha_j v_j\right) = \sum_{i=0}^n \sum_{j=0}^n \alpha_j g_i(v_j) = \sum_{i=0}^n \alpha_i = 1. \qquad \square$$

Proof of Proposition 3.3. Denote $\Delta := \bigcap_{i=0}^n \{x \mid f_i(x) \geqslant 0\}$ which is an n-simplex with vertices, say, $v_0,\dots,v_n$. Let $g_i \in \Delta^a_{[0,1]}$, $0 \leqslant i \leqslant n$, be such that $g_i(v_j) = \delta_{ij}$. Then it is easy to check that $g_i(x) \leqslant f_i(x)$ for all $x \in K$ and all i since $K \subset \Delta$. Thus, we obtain $\sum_{i=0}^n f_i(x) \geqslant \sum_{i=0}^n g_i(x) = 1$ by Lemma 3.2.

For the equality case, $\sum_{i=0}^n f_i(x) = 1$ for $x \in \mathrm{int}K$ iff $g_i(x) = f_i(x)$ and iff $g_i = f_i$ for all i and so iff $K = \Delta$. $\square$

Corollary 3.11. *Let $K \in \mathcal{K}^n$ and $m \geqslant 1$. Then $\sigma^\circ_m(x) \geqslant 1$ for all $x \in \mathrm{int}K$.*

Proof. For any $x \in \mathrm{int}K$, let $\{f_0,\dots,f_n\} \in \mathcal{C}^\circ_{K,n}$, where $f_i(\cdot) = \langle u_i, \cdot\rangle + b_i$, be a minimizer w.r.t. x, i.e., $\sum_{i=0}^m f_i(x) = \sigma^\circ_m(x)$. By Theorem 3.2, there is a simplicial l-support configuration $\{f_{i_0},\dots,f_{i_l}\} \subset \{f_0,\dots,f_n\}$ $(1 \leqslant l \leqslant \min\{m,n\})$. Therefore, with the help of Proposition 3.3, we obtain that, for any $x \in \mathrm{int}K$,

$$\sigma^\circ_m(x) = \sum_{i=0}^m f_i(x) \geqslant \sum_{k=0}^l f_{i_k}(x) = \sum_{k=0}^l f_{i_k}(\boldsymbol{P}x) \geqslant 1,$$

where $\boldsymbol{P}$ is the orthogonal project from $\mathbb{R}^n$ to $\mathrm{lin}\{u_{i_0},\dots,u_{i_l}\}$. $\square$

4 Properties of the sequence $\{\sigma^\circ_m\}_{m\geqslant 1}$

In this section, we show that the dual measures σ°_m share many nice properties with the mean Minkowski measures.

First we show that, similar to $\{\sigma_m\}_{m\geqslant1}$, the sequence $\{\sigma^\circ_m\}$ is sub-arithmetic.

Theorem 4.1. *For any $K \in \mathcal{K}^n, x \in \mathrm{int}K$ and $m,k \geqslant 1$, we have*

$$\sigma^\circ_{m+k}(x) \leqslant \sigma^\circ_m(x) + k\hat{f}(x), \quad x \in \mathrm{int}K,$$

where $\hat{f} = \hat{f}_x \in K^a_{[0,1]}$ satisfies $\hat{f}(x) = \inf_{f\in K^a_{[0,1]}} f(x)$, or equivalently $\frac{1-\hat{f}(x)}{\hat{f}(x)} = \mathrm{as}_\infty(K,x)$. Moreover, the equality holds for $m = n$ and $k \geqslant 1$, i.e., the sequence $\{\sigma^\circ_{n+k}\}_{k\geqslant1}$ is arithmetic.

Proof. For any $\{f_0,\dots,f_m\} \in \mathcal{C}^\circ_m$, denote by $\{f_0,\dots,f_m,\hat{f},\dots,\hat{f}\}$ the $(m+k)$-support configuration with k copies of $\hat{f}$ added to $\{f_0,\dots,f_m\}$. Then we have

$$\begin{aligned}\sigma^\circ_m(x) + k\hat{f}(x) &= \inf_{\{f_0,\dots,f_m\}\in\mathcal{C}^\circ_m} \sum_{i=0}^m f_i(x) + k\hat{f}(x)\\ &= \inf_{\{f_0,\dots,f_m,\hat{f},\dots,\hat{f}\}\in\mathcal{C}^\circ_{m+k}} \left(\sum_{i=0}^m f_i(x) + k\hat{f}(x)\right)\\ &\geqslant \inf_{\{f_0,\dots,f_{m+k}\}\in\mathcal{C}^\circ_{m+k}} \sum_{i=0}^{m+k} f_i(x) = \sigma^\circ_{m+k}(x).\end{aligned}$$

If $k \geqslant 1$, then, for any $\{f_0, \ldots, f_{n+k}\} \in \mathcal{C}^\circ_{n+k}$, by Helly's theorem we have, say, $\{f_0, \ldots, f_n\} \in \mathcal{C}^\circ_n$. Thus, for $x \in \mathrm{int}K$,

$$\begin{aligned}\sum_{i=0}^{n+k} f_i(x) &= \sum_{i=0}^{n} f_i(x) + \sum_{i=n+1}^{m+k} f_i(x) \\ &\geqslant \sigma^\circ_n(x) + k\hat{f}(x),\end{aligned}$$

which leads clearly to $\sigma^\circ_{n+k}(x) \geqslant \sigma^\circ_n(x) + k\hat{f}(x)$ and in turn (together with the reverse inequality just proved above) leads to $\sigma^\circ_{n+k}(x) = \sigma^\circ_n(x) + k\hat{f}(x)$, i.e., the sequence $\{\sigma^\circ_{n+k}\}_{k\geqslant 1}$ is arithmetic. □

We now show that, similar to $\{\sigma_m\}$, the sequence $\{\sigma^\circ_m\}$ is also upper-additive.

Theorem 4.2. *Let $K \in \mathcal{K}$ and $m, k \geqslant 1$. Then we have*

$$\sigma^\circ_{m+k} - \sigma^\circ_{m+1} \geqslant \sigma^\circ_k - \sigma^\circ_1.$$

Proof. Let $\{f_0, \ldots, f_{m+k}\} \in \mathcal{C}^\circ_{m+k}$ be an $(m+k)$-minimizer w.r.t. $x \in \mathrm{int}K$.

If $\{f_0, \ldots, f_m\} \in \mathcal{C}^\circ_m$, then by applying Theorem 4.1 repeatedly, we have

$$\begin{aligned}\sigma^\circ_{m+k}(x) &= \sum_{i=0}^{m} f_i(x) + \sum_{i=m+1}^{m+k} f_i(x) \\ &\geqslant \sigma^\circ_m(x) + k\hat{f}(x) \\ &\geqslant \sigma^\circ_{m+1}(x) + (k-1)\hat{f}(x) \\ &= \sigma^\circ_{m+1}(x) + (\sigma^\circ_1(x) + (k-1)\hat{f}(x)) - \sigma^\circ_1(x) \\ &\geqslant \sigma^\circ_{m+1}(x) + \sigma^\circ_k(x) - \sigma^\circ_1(x),\end{aligned}$$

where $\hat{f}$ is the same as in Theorem 4.1.

If $\{f_{m+1}, \ldots, f_{m+k}\} \in \mathcal{C}^\circ_{k-1}$, we have, by applying Theorem 4.1 repeatedly again,

$$\begin{aligned}\sigma^\circ_{m+k}(x) &= \sum_{i=0}^{m} f_i(x) + \sum_{i=m+1}^{m+k} f_i(x) \\ &\geqslant (m+1)\hat{f}(x) + \sigma^\circ_{k-1}(x) \\ &= (m\hat{f}(x) + \sigma^\circ_1(x)) + (\hat{f}(x) + \sigma^\circ_{k-1}(x)) - \sigma^\circ_1(x) \\ &\geqslant \sigma^\circ_{m+1}(x) + \sigma^\circ_k(x) - \sigma^\circ_1(x).\end{aligned}$$

Now, suppose that $\{f_0, \ldots, f_m\} \notin \mathcal{C}^\circ_m$ and $\{f_{m+1}, \ldots, f_{m+k}\} \notin \mathcal{C}^\circ_{k-1}$. We observe first that, by Theorem 3.1, there are nonnegative α_i, $i = 0, 1, \ldots, m+k$, with $\alpha_{i_0} > 0$ for at least one i_0 such that $\sum_{i=0}^{m+k} \alpha_i u_i = o$, from which we get $\sum_{i=0}^{m} \alpha_i u_i = -\sum_{i=m+1}^{m+k} \alpha_i u_i =: u$. We claim that $u \neq o$ (so not all α_i, $i = 0, \ldots, m$, are zero and not all α_i, $i = m+1, \ldots, m+k$, are zero) for otherwise we would have by Theorem 3.1 again $\{f_0, \ldots, f_m\} \in \mathcal{C}^\circ_m$ or $\{f_{m+1}, \ldots, f_{m+k}\} \in \mathcal{C}^\circ_{k-1}$.

We set $f(\cdot) := \langle u, \cdot\rangle + b \in K^a_{[0,1]}$ (and so $1 - f = \langle -u, \cdot\rangle + 1 - b \in K^a_{[0,1]}$ as well), where $b = \sum_{i=0}^{m} \alpha_i b_i$. Thus, it is easy to check that by Lemma 3.1 $\bigcap_{i=0}^{m}\{f_i \leqslant 0\} \cap \{1 - f \leqslant 0\} = \bigcap_{i=0}^{m}\{f_i \leqslant 0\} = \emptyset$, i.e., $\{f_0, \ldots, f_m, 1-f\} \in \mathcal{C}^\circ_{m+1}$. Similarly, $\{f_{m+1}, \ldots, f_{m+k}, f\} \in \mathcal{C}^\circ_k$. Thus,

$$\begin{aligned}\sigma^\circ_{m+k}(x) &= \sum_{i=0}^{m} f_i(x) + \sum_{i=m+1}^{m+k} f_i(x) \\ &= \left(\sum_{i=0}^{m} f_i(x) + (1 - f(x))\right) + \left(\sum_{i=m+1}^{m+k} f_i(x) + f(x)\right) - 1 \\ &\geqslant \sigma^\circ_{m+1}(x) + \sigma^\circ_k(x) - \sigma^\circ_1(x),\end{aligned}$$

where we used the fact $\sigma^\circ_1 \equiv 1$. □

The following are some applications of Theorem 4.1: The first one reveals the relation between the Minkowski measure of asymmetry and the dual mean Minkowski measure of symmetry and the second one implies that the concave function $\sigma_m^\circ : \mathrm{int}K \to \mathbb{R}$ can be continuously extended to the whole K by setting it equal to 1 on $\boldsymbol{bd}K$.

Proposition 4.3. *Let* $K \in \mathcal{K}^n$. *Then* $\lim_{m\to\infty} \frac{\sigma_m^\circ(x)}{m} = \frac{1}{1+\mathrm{as}_\infty(x)}$.

Proof. By Theorem 4.1 and the fact that $\sigma_m^\circ(x) \geqslant (m+1)\hat{f}(x)$, where $\hat{f}$ is the same as in Theorem 4.1, we have

$$\begin{aligned}\hat{f}(x) &= \lim_{m\to\infty} \frac{m+1}{m}\hat{f}(x) \leqslant \lim_{m\to\infty} \frac{\sigma_m^\circ(x)}{m} = \lim_{m\to\infty} \frac{\sigma_{1+(m-1)}^\circ(x)}{m} \\ &\leqslant \lim_{m\to\infty} \frac{\sigma_1^\circ(x) + (m-1)\hat{f}(x)}{m} = \hat{f}(x).\end{aligned}$$

So $\lim_{m\to\infty} \frac{\sigma_m^\circ(x)}{m} = \hat{f}(x) = \frac{1}{1+\mathrm{as}_\infty(x)}$. □

Proposition 4.4. *Let* $K \in \mathcal{K}^n$ *and* $x_0 \in \boldsymbol{bd}K$. *Then* $\lim_{x\to x_0} \sigma_m^\circ(x) = 1$.

Proof. By Corollary 3.3 and Theorem 4.1, we have

$$\begin{aligned}1 &\leqslant \lim_{x\to x_0} \sigma_m^\circ(x) \leqslant \lim_{x\to x_0} (\sigma_1^\circ(x) + (m-1)\hat{f}_x(x)) \\ &= \lim_{x\to x_0} (1 + (m-1)\hat{f}_x(x)) = 1,\end{aligned}$$

where we used the obvious facts that $\sigma_1^\circ \equiv 1$ and that $\lim_{x\to x_0} \hat{f}_x(x) = 0$. Therefore $\lim_{x\to x_0} \sigma_m^\circ(x) = 1$. □

5 Proof of main theorem

We start this section with two lemmas which will be needed for the proof of Theorem B.

Lemma 5.1. *If* $\sigma_n^\circ(x) = 1$ *for some* $x \in \mathrm{int}K$ *and there is a simplicial* n*-minimizer w.r.t.* x, *then* K *is an* n*-simplex* (*and so* $\sigma_n^\circ \equiv 1$).

Proof. Suppose $\sigma_n^\circ(x) = 1$ for some $x \in \mathrm{int}K$ and $\{f_0, \ldots, f_n\} \in \mathcal{C}_n^\circ$ is a simplicial minimizer w.r.t. x, i.e. $\Delta := \bigcap_{i=0}^n \{f_i \geqslant 0\}$ is an n-simplex. For $0 \leqslant i \leqslant n$, let $g_i \in \Delta_{[0,1]}^a$ be such that $\{g_i = 0\} = \{f_i = 0\}$, then $g_i(x) \leqslant f_i(x)$ (since $K \subset \Delta$).

If K is not an n-simplex, then there exists at least one i such that $g_i(x) < f_i(x)$. Thus $\sigma_n^\circ(x) = \sum_{i=0}^n f_i(x) > \sum_{i=0}^n g_i(x) = 1$ (where the last equality follows from Lemma 3.2), a contradiction! □

Lemma 5.2. *If* $\sigma_m^\circ(x) = 1$ $(m \leqslant n)$ *for some* $x \in \mathrm{int}K$, *then all* m*-minimizers w.r.t.* x *are simplicial.*

Proof. If $\{f_0, \ldots, f_m\} \in \mathcal{C}_m^\circ$ is not m-simplicial, then by Theorem 3.2 and Proposition 3.2, $\{f_0, \ldots, f_m\}$ has a proper sub-support configuration, say, $\{f_0, \ldots, f_l\}$, where $l < m$. Thus we have

$$\sum_{i=0}^m f_i(x) = \sum_{k=0}^l f_i(x) + \sum_{i=l+1}^m f_i(x) > \sigma_l^\circ(x) \geqslant 1.$$

So $\{f_0, \ldots, f_m\}$ is not an m-minimizer w.r.t. x. □

Now, it is the time to prove Theorem B.

Proof of Theorem B. First, we consider the inequalities. Since $\sigma_m^\circ \geqslant 1$ was already shown in Corollary 3.3, we need only to show $\sigma_m^\circ \leqslant \frac{m+1}{2}$.

By Theorem 4.1,

$$\sigma_m^\circ(x) = \sigma_{1+(m-1)}^\circ(x) \leqslant \sigma_1^\circ(x) + (m-1)\hat{f}(x) \leqslant 1 + \frac{m-1}{2} = \frac{m+1}{2},$$

where we used the fact that $\hat{f}(x) \leqslant \frac{1}{2}$ for all $x \in \mathrm{int}K$ (which can be easily seen from the definition of $\hat{f}$).

We now discuss the equality cases.

Suppose that K is a symmetric body centered at x, then $f(x) = \frac{1}{2}$ for all $f \in K^a_{[0,1]}$ and so $\sum_{i=0}^m f_i(x) = \frac{m+1}{2}$ for any $\{f_0, \ldots, f_m\} \in \mathcal{C}^\circ_m$, and in turn $\sigma^\circ_m(x) = \frac{m+1}{2}$.

Conversely, if $\sigma^\circ_m(x) = \frac{m+1}{2}$ for some $x \in \mathrm{int}K$, then by Theorem 4.1,

$$\frac{m+1}{2} = \sigma^\circ_m(x) \leqslant \sigma^\circ_{m-1}(x) + \hat{f}(x) \leqslant \frac{m}{2} + \frac{1}{2} = \frac{m+1}{2},$$

where we used the fact that $\sigma^\circ_{m-1}(x) \leqslant \frac{m}{2}$ and $\hat{f}(x) \leqslant \frac{1}{2}$, which implies clearly $\hat{f}(x) = \frac{1}{2}$. Thus, by the definition of $\hat{f}$, we have $f(x) = \frac{1}{2}$ for all $f \in K^a_{[0,1]}$ and in turn that K is a symmetric body centered at x.

Now suppose there is an orthogonal project $P_H : \mathbb{R}^n \to H$, an m-dimensional subspace ($2 \leqslant m \leqslant n$), such that $\Delta := P_H(K)$ is an m-simplex in H with vertices, say, $v_0, \ldots, v_m$. Let affine functions $\tilde{f}_i : H \to \mathbb{R}$ ($0 \leqslant i \leqslant m$) be such that $\tilde{f}_i(v_j) = \delta_{ij}$. Then $\{\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_m\} \in \mathcal{C}^\circ_{\Delta,m}$ and $\sum_{i=0}^m \tilde{f}_i \equiv 1$ in $\mathrm{int}\Delta$. Now, setting $f_i := \tilde{f}_i \circ P_H : \mathbb{R}^n \to \mathbb{R}$ ($0 \leqslant i \leqslant m$), we have clearly $\{f_0, \ldots, f_m\} \in \mathcal{C}^\circ_{K,m}$ and $\sum_{i=0}^m f_i(x) = \sum_{i=0}^m \tilde{f}_i(P_H x) = 1$ for $x \in \mathrm{int}K$, i.e., $\sigma^\circ_m \equiv 1$.

Conversely, if $\sigma^\circ_m(x) = 1$ for some $x \in \mathrm{int}K$ and $\{f_0, \ldots, f_m\} \in \mathcal{C}^\circ_m$ is a m-minimizer w.r.t. x, i.e., $\sum_{i=0}^m f_i(x) = 1$, then $m \leqslant n$ for otherwise, by Helly's theorem there are, say, $f_0, \ldots, f_n$ such that $\{f_0, \ldots, f_n\} \in \mathcal{C}^\circ_n$, and so we would have $1 = \sigma^\circ_m(x) = \sum_{i=0}^m f_i(x) = \sum_{i=0}^n f_i(x) + \sum_{i=n+1}^m f_i(x) \geqslant \sigma^\circ_n(x) + \sum_{i=n+1}^m f_i(x) > 1$.

By Lemma 5.2, $\{f_0, \ldots, f_m\}$ is m-simplicial. Now setting $H := \mathrm{lin}\{u_i\}_{i=0}^m$ where $u_0, \ldots, u_m$ are such that $f_i(\cdot) = \langle u_i, \cdot \rangle + b_i$, we have, by the definition of simplicial support configurations, that $P_H(K)$ is an m-simplex in H.

The proof is complete. □

6 Conclusions and further considerations

As mentioned in Introduction and shown in later sections, in contrast to the mean Minkowski measures which describe the shapes of low-dimensional sections of a convex body, the dual mean Minkowski measures of symmetry provide indeed some valuable information on low-dimensional orthogonal projections of a convex body, in particular, on low-dimensional simplicial projections. To the best of our knowledge, the mean Minkowski and the dual mean Minkowski measures of symmetry are probably the only measures which provide information not only on the shape of a convex body itself but also on the shapes of its sections or projections. We expect more such kinds of measures of symmetry (or asymmetry) to be found.

In this paper, we pay attention mainly to the best lower/upper bounds of dual mean Minkowski measures and on determining the corresponding extremal projections. There are still more problems to be studied, e.g., the stabilities of dual mean Minkowski measures at both the extremal values, i.e., 1 and $\frac{m+1}{2}$; the properties of the set of σ°_m-critical points. Also, we think it hopeful that Grünbaum's conjecture is valid at a σ°_n-critical point, i.e., there should be $n+1$ affine diameters meeting at one σ°_n-critical point of a convex body (*cf.* [7]).

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11271282) and the Jiangsu Specified Fund for Foreigner Scholars 2014–2015. The authors thank the referees for their carefully reading the manuscript and pointing out some typos.

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