Optimal shape and placement of actuators or sensors for PDE models

Y. Privat, E. Trélat\textsuperscript{1}, E. Zuazua

\textsuperscript{1}Univ. Paris 6 (Labo. J.-L. Lions) et Institut Universitaire de France

Rutgers University - Camden, March 15th, 2013

Kinetic Interaction Team
Observation of wave equations

- \((M, g)\) Riemannian manifold
- \(\triangle_g\) Laplace-Beltrami
- \(\Omega\) open bounded connected subset of \(M\)
- \(\omega \subset \Omega\) subset of positive measure

Wave equation

\[
y_{tt} = \triangle_g y, \quad (t, x) \in (0, T) \times \Omega,
\]
\[
y(0, \cdot) = y^0 \in L^2(\Omega), \quad y_t(0, \cdot) = y^1 \in H^{-1}(\Omega)
\]

Schrödinger equation

\[
iy_t = \triangle_g y, \quad (t, x) \in (0, T) \times \Omega,
\]
\[
y(0, \cdot) = y^0 \in L^2(\Omega)
\]

If \(\partial \Omega \neq \emptyset\), then Dirichlet boundary conditions: \(y(t, \cdot)|_{\partial \Omega} = 0\).
(also: Neumann, mixed, or Robin boundary conditions)

Observable

\[
z = \chi_\omega y
\]
Observability

**Observability inequality**

The system is said observable (in time $T > 0$) if there exists $C_T(\omega) > 0$ such that

$$
\forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad C_T(\omega) \| (y^0, y^1) \|^2_{L^2 \times H^{-1}} \leq \int_0^T \int_\omega y(t, x)^2 \, dx \, dt.
$$

Bardos-Lebeau-Rauch (1992): the observability inequality holds if the pair $(\omega, T)$ satisfies the Geometric Control Condition (GCC) in $\Omega$:

*Every ray of geometrical optics that propagates in $\Omega$ and is reflected on its boundary $\partial \Omega$ intersects $\omega$ in time less than $T$.***
Observability

Observability inequality

The system is said observable (in time $T > 0$) if there exists $C_T(\omega) > 0$ such that

$$\forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega), \quad C_T(\omega)\| (y^0, y^1) \|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_{\omega} y(t, x)^2 \, dx \, dt.$$  

Bardos-Lebeau-Rauch (1992): the observability inequality holds if the pair $(\omega, T)$ satisfies the Geometric Control Condition (GCC) in $\Omega$:

*Every ray of geometrical optics that propagates in $\Omega$ and is reflected on its boundary $\partial \Omega$ intersects $\omega$ in time less than $T$.*

Question

What is the "best possible" control domain $\omega$ of fixed given measure?
1) What is the "best domain" for achieving HUM optimal control?

\[ y_{tt} - \Delta y = \chi \omega u \]

2) What is the "best domain" domain for stabilization (with localized damping)?

\[ y_{tt} - \Delta y = -k \chi \omega y_t \]

See works by
- P. Hébrard, A. Henrot: theoretical and numerical results in 1D for optimal stabilization (for all initial data).
- S. Cox, P. Freitas, F. Fahroo, K. Ito, ...: variational formulations and numerics.
- M.I. Frecker, C.S. Kubrusly, H. Malebranche, S. Kumar, J.H. Seinfeld, ...: numerical investigations (among a finite number of possible initial data).
- ...
Let $L \in (0, 1)$ and $T > 0$ fixed.

It is a priori natural to model the problem as:

Uniform optimal design problem

Maximize

$$C_T(\omega) = \inf \left\{ \frac{\int_0^T \int_{\omega} y(t, x)^2 \, dx \, dt}{\| (y^0, y^1) \|_{L^2 \times H^{-1}}^2} \mid (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \setminus \{(0, 0)\} \right\}$$

or such a kind of criterion, over all possible subsets $\omega \subset \Omega$ of measure $|\omega| = L|\Omega|$.

BUT...
Two difficulties arise with this model.

1. Theoretical difficulty.
2. The model is not relevant wrt practice.
Two difficulties arise with this model.

1. **Theoretical difficulty.**

2. The model is not relevant wrt practice.
Every solution can be expanded as

\[ y(t, x) = \sum_{j=1}^{+\infty} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)) \phi_j(x) \]

with \( a_j = \int_{\Omega} y^0(x) \phi_j(x) \, dx \), \( b_j = \frac{1}{\lambda_j} \int_{\Omega} y^1(x) \phi_j(x) \, dx \), for every \( j \in \mathbb{N}^* \). Moreover, \( \| (y^0, y^1) \|_{L^2 \times H^{-1}}^2 = \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \). Then:

\[
\int_0^T \int_\omega y(t, x)^2 \, dx \, dt = \int_0^T \int_\omega \left( \sum_{j=1}^{+\infty} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)) \phi_j(x) \right)^2 \, dx \, dt \\
= \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_\omega \phi_i(x) \phi_j(x) \, dx
\]

where \( \alpha_{ij} = \int_0^T (a_i \cos(\lambda_i t) + b_i \sin(\lambda_i t))(a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)) \, dt \).

The coefficients \( \alpha_{ij} \) depend only on the initial data \((y^0, y^1)\).
Spectral expansion

Conclusion:

$$
\int_0^T \int_\omega y(t, x)^2 \, dx \, dt = \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_\omega \phi_i(x) \phi_j(x) \, dx
$$

with

$$
\alpha_{ij} = \begin{cases} 
    a_i a_j \left( \frac{\sin(\lambda_i + \lambda_j) T}{2(i+j)} + \frac{\sin(\lambda_i - \lambda_j) T}{2(\lambda_i - \lambda_j)} \right) + a_i b_j \left( \frac{1 - \cos(\lambda_i + \lambda_j) T}{2(\lambda_i + \lambda_j)} - \frac{1 - \cos(\lambda_i - \lambda_j) T}{2(\lambda_i - \lambda_j)} \right) \\
    + a_j b_i \left( \frac{1 - \cos(\lambda_i + \lambda_j) T}{2(\lambda_i + \lambda_j)} + \frac{1 - \cos(\lambda_i - \lambda_j) T}{2(\lambda_i - \lambda_j)} \right) + b_i b_j \left( -\frac{\sin(\lambda_i + \lambda_j) T}{2(\lambda_i + \lambda_j)} + \frac{\sin(\lambda_i - \lambda_j) T}{2(\lambda_i - \lambda_j)} \right) & \text{if } \lambda_i \neq \lambda_j, \\
    a_j^2 \left( \frac{T}{2} + \frac{\sin 2\lambda_j T}{4\lambda_j} \right) + a_j b_j \left( \frac{1 - \cos 2\lambda_j T}{2\lambda_j} \right) + b_j^2 \left( \frac{T}{2} - \frac{\sin 2\lambda_j T}{4\lambda_j} \right) & \text{if } \lambda_i = \lambda_j.
\end{cases}
$$

The coefficients $\alpha_{ij}$ depend only on the initial data $(y^0, y^1)$. 
Solving of the uniform design problem

\[ \sup_{\omega \subset \Omega} \left( \frac{\omega}{L|\Omega|} \right) = \sup_{\omega \subset \Omega} \left( \frac{\omega}{L|\Omega|} \right) \inf_{\sum(a_j^2 + b_j^2) = 1} \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_\omega \phi_i(x) \phi_j(x) \, dx \]

→ serious difficulty due to the crossed terms.

Same difficulty in the (open) problem of determining the optimal constants in Ingham’s inequalities.
Two difficulties arise with this model.

1. Theoretical difficulty.

2. The model is not relevant wrt practical expectation.
The usual observability constant is deterministic and gives an account for the worst case. It is **pessimistic**.

In practice: many experiments, many measures.

**Objective**: optimize the sensor shape and location *in average*.

→ **randomized** observability constant.
Randomized observability constant

Averaging over random initial data:

Randomized observability inequality

\[ C_{T, \text{rand}}(\omega) \|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \mathbb{E} \left( \int_0^T \int_\omega y_\nu(t, x)^2 \, dx \, dt \right) \]

where \( y_\nu(t, x) = \sum_{j=1}^{+\infty} \left( \beta_{1,j}^\nu a_j e^{i\lambda_j t} + \beta_{2,j}^\nu b_j e^{-i\lambda_j t} \right) \phi_j(x) \), with \( \beta_{1,j}^\nu, \beta_{2,j}^\nu \) i.i.d. Bernoulli.


Theorem

\[ C_{T, \text{rand}}(\chi_\omega) = \frac{T}{2} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx. \]

Remark

There holds \( C_{T, \text{rand}}(\chi_\omega) \geq C_T(\chi_\omega) \). There are examples where the inequality is strict.
Conclusion: we model the problem as

$$\sup_{\omega \subset \Omega \quad |\omega| = L |\Omega|} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx$$

Remark

This is an energy (de)concentration criterion.
Remark: another way of arriving at this criterion

**Averaging in time:**
Time asymptotic observability inequality:

\[
C_\infty(\chi_\omega) \|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt,
\]

with

\[
C_\infty(\chi_\omega) = \inf \left\{ \lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt \middle| (y^0, y^1) \in L^2 \times H^{-1} \setminus \{(0,0)\} \right\}.
\]

**Theorem**

If the eigenvalues of $\triangle g$ are simple then

\[
C_\infty(\chi_\omega) = \frac{1}{2} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx.
\]

**Remarks**

- \(C_\infty(\chi_\omega) \leq \frac{1}{2} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx.\)
- \(\limsup_{T \to +\infty} \frac{C_T(\chi_\omega)}{T} \leq C_\infty(\chi_\omega).\) There are examples where the inequality is strict.
Solving of the problem

\[
\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\Omega} \chi_{\omega}(x) \phi^2_j(x) \, dx
\]

1. Convexification procedure

\[
\overline{U}_L = \{ a \in L^\infty(\Omega, (0, 1)) \mid \int_{\Omega} a(x) \, dx = L|\Omega| \}.
\]

\[\longrightarrow \sup_{a \in \overline{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi^2_j(x) \, dx\]

A priori:

\[
\sup_{\omega \subset \Omega, \, |\omega| = L|\Omega|} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi^2_j(x) \, dx \leq \sup_{a \in \overline{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi^2_j(x) \, dx.
\]
Solving of the problem

Moreover, under the assumptions

**WQE (weak Quantum Ergodicity) Assumption**

There exists a subsequence such that

$$\phi_j^2 \, dx \rightharpoonup \frac{dx}{|\Omega|} \text{ vaguely.}$$

we have

$$\sup_{a \in \overline{U_L}} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j^2(x) \, dx = L$$

(reached with \(a \equiv L\))

**Uniform \(L^\infty\)-boundedness**

There exists \(A > 0\) such that

$$\|\phi_j\|_{L^\infty} \leq A.$$  

**Remarks**

- It is true in 1D, since \(\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)\) on \(\Omega = [0, \pi]\).
  Moreover, this relaxed problem has an infinite number of solutions, given by

$$a(x) = L + \sum_j (a_j \cos(2jx) + b_j \sin(2jx)) \text{ with } a_j \leq 0$$

(and with \(|a_j|\) and \(|b_j|\) small enough so that \(0 \leq a(\cdot) \leq 1\)).
Solving of the problem

Moreover, under the assumptions

**WQE (weak Quantum Ergodicity) Assumption**

There exists a subsequence such that

\[ \phi_j^2 \, dx \rightharpoonup \frac{dx}{|\Omega|} \text{ vaguely.} \]

we have

\[ \sup_{a \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^+} \int_{\Omega} a(x) \phi_j^2(x) \, dx = L \]

(reached with \( a \equiv L \))

**Uniform \( L^\infty \)-boundedness**

There exists \( A > 0 \) such that

\[ \| \phi_j \|_{L^\infty} \leq A. \]

**Remarks**

- In multi-D:
  - \( L^\infty \)-WQE is true in any flat torus.
  - If \( \Omega \) is an ergodic billiard with \( W^{2,\infty} \) boundary then
    \[ \phi_j^2 \, dx \rightharpoonup \frac{1}{|\Omega|} \, dx \]
    vaguely for a subset of indices of density 1.

(see also Shnirelman, Burq-Zworski, Colin de Verdière, ...)

E. Trélat

Optimal shape and placement of actuators or sensors
Solving of the problem

2. Gap or no-gap?

A priori, under WQE and uniform $L^\infty$ boundedness assumptions:

$$\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j^2(x) \, dx \leq \sup_{a \in \mathcal{A}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j^2(x) \, dx = L.$$

Remarks in 1D:

- Note that, for every $\omega$, $\frac{2}{\pi} \int_{\omega} \sin^2(jx) \, dx \rightarrow L$ as $j \rightarrow +\infty$.
- No lower semi-continuity property of the criterion.
- With $\omega_N = \bigcup_{k=1}^{N} \left[ \frac{k\pi}{N+1}, \frac{k\pi}{N+1} + \frac{L\pi}{2N} \right]$, one has $\chi_{\omega_N} \rightharpoonup L$ but

$$\lim_{N \rightarrow +\infty} \inf_{j \in \mathbb{N}^*} \frac{2}{\pi} \int_{\omega_N} \sin^2(jx) \, dx < L.$$

$\Rightarrow$ this cannot follow from usual $\Gamma$-convergence arguments.
Solving of the problem

Theorem 1

Under WQE and uniform $L^\infty$ boundedness assumptions, there is no gap, that is:

$$\sup_{\chi_{\omega} \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} \chi_{\omega}(x) \phi_j(x)^2 \, dx = \sup_{a \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j(x)^2 \, dx = L.$$ 

→ the maximal value of the time-asymptotic / randomized observability constant is $L$.

Remark

The assumptions are sufficient but not sharp: the result also holds also true in the Euclidean disk, for which however the eigenfunctions are not uniformly bounded in $L^\infty$ (whispering galleries phenomenon).
Theorem 1

Under WQE and uniform $L^\infty$ boundedness assumptions, there is no gap, that is:

$$\sup_{\chi_{\omega} \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} \chi_{\omega}(x) \phi_j(x)^2 \, dx = \sup_{a \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j(x)^2 \, dx = L.$$  

→ the maximal value of the time-asymptotic / randomized observability constant is $L$.

Remark

The assumptions are sufficient but not sharp: the result also holds true in the Euclidean disk, for which however the eigenfunctions are not uniformly bounded in $L^\infty$ (whispering galleries phenomenon).
Solving of the problem

**QUE (Quantum Unique Ergodicity) Assumption**

We assume that \( \phi_j^2 \, dx \rightarrow \frac{dx}{|\Omega|} \) vaguely. (i.e. the whole sequence converges to the uniform measure)

\[
\mathcal{U}_L^b = \{ \chi_\omega \in \mathcal{U}_L \mid |\partial \omega| = 0 \}
\]

**Theorem 2**

Under QUE + \( L^p \)-boundedness of the \( \phi_j \)'s for some \( p > 2 \),

\[
\sup_{\chi_\omega \in \mathcal{U}_L^b} \inf_{j \in \mathbb{N}^*} \int_\Omega \chi_\omega(x) \phi_j(x)^2 \, dx = L.
\]

Remark: The result holds as well if one replaces \( \mathcal{U}_L^b \) with either the set of open subsets having a Lipschitz boundary, or with a bounded perimeter.
Solving of the problem

Comments on ergodicity assumptions:

- true in 1D, since \( \phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx) \) on \( \Omega = [0, \pi] \).
- **Quantum Ergodicity property** (QE) in multi-D:
    If \( \Omega \) is an ergodic billiard with \( W^{2,\infty} \) boundary then \( \phi_j^2 \, dx \rightharpoonup \frac{dx}{|\Omega|} \) vaguely for a subset of indices of density 1.
  - Strictly convex billiards sufficiently regular are not ergodic (Lazutkin, 1973).
    Rational polygonal billiards are not ergodic.
    Generic polygonal billiards are ergodic (Kerckhoff-Masur-Smillie, Ann. Math. ’86).
  - There exist some convex sets \( \Omega \) (stadium shaped) that satisfy QE but not QUE (Hassell, Ann. Math. 2010)
  - QUE conjecture (Rudnick-Sarnak 1994): every compact manifold having negative sectional curvature satisfies QUE.
Solving of the problem

Hence in general this assumption is related with ergodic / concentration / entropy properties of eigenfunctions.

See Shnirelman, Sarnak, Bourgain-Lindenstrauss, Colin de Verdière, Anantharaman, Nonnenmacher, De Bièvre, ...

If this assumption fails, we may have scars: energy concentration phenomena (there can be exceptional subsequences converging to other invariant measures, like, for instance, measures carried by closed geodesics: scars)
Solving of the problem

stable trapped ray

instable trapped ray
Come back to the theorem:

Under certain quantum ergodicity assumptions, there holds

$$\sup_{\chi \omega \in \mathcal{U}} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx = L.$$ 

Moreover:
We are able to prove that, for certain sets $\Omega$, the second problem does not have any solution (i.e., the supremum is not reached).
We conjecture that this property is generic.
Solving of the problem

Last remark:

The proof of this no-gap result is based on a quite technical homogenization-like procedure. In dimension one, it happens that it is equivalent to the following harmonic analysis result:

Let $\mathcal{F}$ the set of functions

$$f(x) = L + \sum_{j=1}^{+\infty} (a_j \cos(2jx) + b_j \sin(2jx)), \quad \text{with } a_j \leq 0 \quad \forall j \in \mathbb{N}^*.$$  

Then:

$$d(\mathcal{F}, \mathcal{U}_L) = 0$$

but there is no $\chi_\omega \in \mathcal{F}$.

(Where $\mathcal{U}_L = \{\chi_\omega \mid \omega \subset [0, \pi], |\omega| = L\pi\}$)
Since the second problem may have no solution, it makes sense to consider as in


a truncated version of the second problem:

$$\sup_{\omega \subset \Omega} \min_{1 \leq j \leq N} \int_{\omega} \phi_j^2(x) \, dx$$
Theorem

The problem has a unique solution $\omega^N$. Moreover, $\omega^N$ has a finite number of connected components. If $\Omega$ has a symmetry hyperplane, then $\omega^N$ enjoys the same symmetry property.
Truncated version of the second problem

Theorem, specific to the 1D case

$\omega^N$ is symmetric with respect to $\pi/2$, is the union of at most $N$ intervals, and:

there exists $L_N \in (0, 1]$ such that, for every $L \in (0, L_N]$,

$$\int_{\omega^N} \sin^2 x \, dx = \int_{\omega^N} \sin^2(2x) \, dx = \cdots = \int_{\omega^N} \sin^2(Nx) \, dx.$$  

Equality of the criteria $\Rightarrow$ the optimal domain $\omega^N$ concentrates around the points $\frac{k\pi}{N+1}$, $k = 1, \ldots, N$.

Spillover phenomenon: the best domain $\omega^N$ for the $N$ rst modes is the worst possible for the $N + 1$ first modes.
Several numerical simulations

Problem 2 (Dirichlet case): Optimal domain for $N=2$ and $L=0.2$

Problem 2 (Dirichlet case): Optimal domain for $N=5$ and $L=0.2$

Problem 2 (Dirichlet case): Optimal domain for $N=10$ and $L=0.2$

Problem 2 (Dirichlet case): Optimal domain for $N=20$ and $L=0.2$

Problem 2 (Dirichlet case): Optimal domain for $N=2$ and $L=0.4$

Problem 2 (Dirichlet case): Optimal domain for $N=5$ and $L=0.4$

Problem 2 (Dirichlet case): Optimal domain for $N=10$ and $L=0.4$

Problem 2 (Dirichlet case): Optimal domain for $N=20$ and $L=0.4$

Problem 2 (Dirichlet case): Optimal domain for $N=2$ and $L=0.6$

Problem 2 (Dirichlet case): Optimal domain for $N=5$ and $L=0.6$

Problem 2 (Dirichlet case): Optimal domain for $N=10$ and $L=0.6$

Problem 2 (Dirichlet case): Optimal domain for $N=20$ and $L=0.6$

E. Trélat

Optimal shape and placement of actuators or sensors
Several numerical simulations

Problem 2 (Dirichlet case): Optimal domain for $N=1$ and $L=0.2$

Problem 2 (Dirichlet case): Optimal domain for $N=2$ and $L=0.2$

Problem 2 (Dirichlet case): Optimal domain for $N=5$ and $L=0.2$

Problem 2 (Dirichlet case): Optimal domain for $N=10$ and $L=0.2$

Problem 2 (Dirichlet case): Optimal domain for $N=20$ and $L=0.2$

E. Trélat

Optimal shape and placement of actuators or sensors
Further comments

1. Existence of a maximizer

Ensured if $\mathcal{U}_L$ is replaced with any of the following choices:

- $\mathcal{V}_L = \{ \chi_\omega \in \mathcal{U}_L \mid P_\Omega(\omega) \leq A \}$ (perimeter)
- $\mathcal{V}_L = \{ \chi_\omega \in \mathcal{U}_L \mid \| \chi_\omega \|_{BV(\Omega)} \leq A \}$ (total variation)
- $\mathcal{V}_L = \{ \chi_\omega \in \mathcal{U}_L \mid \omega \text{ satisfies the } 1/A\text{-cone property} \}$

where $A > 0$ is fixed.
2. Intrinsic variant.

Maximize

\[
\inf_{\phi \in \mathcal{E}} \int_{\omega} \phi(x)^2 \, dx
\]

over all possible subsets \( \omega \) of \( \Omega \) of measure \( |\omega| = L|\Omega| \), where \( \mathcal{E} \) denotes the set of all normalized eigenfunctions of \( \triangle g \).

→ Same no-gap results as before.

→ Examples of gap: unit sphere of \( \mathbb{R}^3 \), or half-unit sphere with Dirichlet boundary conditions.
3. Parabolic equations

\[ y_t = Ay \]

(for example, heat equation)

Observability inequality:

\[ C_T(\chi_\omega) \|y(T, \cdot)\|_{L^2}^2 \leq \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt \]

→ The situation is very different.
Further comments

3. Parabolic equations

\[ y_t = Ay \]

(for example, heat equation)

In that case, the problem is reduced (by averaging either in time or w.r.t. random initial conditions) to

\[ \sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega |\phi_j(x)|^2 \, dx \quad \text{with} \quad \gamma_j = \frac{e^{2\Re(\lambda_j)T} - 1}{2\Re(\lambda_j)}. \]

**Theorem**

Assume that

\[ \liminf_{j \to +\infty} \gamma_j(T) \int_\Omega a(x)|\phi_j(x)|^2 \, dx > \gamma_1(T), \]

for every \( a \in \overline{\mathcal{U}}_L \). Then there exists \( N \in \mathbb{N}^* \) such that

\[ \sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega |\phi_j|^2 = \max_{\chi_\omega \in \mathcal{U}_L} \inf_{1 \leq j \leq n} \gamma_j \int_\omega |\phi_j|^2, \]

for every \( n \geq N \). In particular there is a unique solution \( \chi_{\omega^N} \). Moreover if \( M \) is analytic then \( \omega^N \) is semi-analytic and has a finite number of connected components.
$L = 0.2, T = 0.05, \Omega = [0, \pi]^2$, Dirichlet boundary conditions.

$N \in \{1, 2, 3, 4, 5, 6\}$.
Conclusion and perspectives

- Same kind of analysis for the optimal design of the control domain.
- Intimate relations between domain optimization and quantum chaos (quantum ergodicity properties).

Next issues (ongoing works with Y. Privat and E. Zuazua)

- Consider other kinds of spectral criteria permitting to avoid spillover.
- Discretization issues: do the numerical optimal designs converge to the continuous optimal design as the mesh size tends to 0?

Y. Privat, E. Trélat, E. Zuazua,

- *Optimal shape and location of sensors or controllers for parabolic equations with random initial data*, Preprint.