Optimal shape and placement of actuators or sensors for PDE models

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Kinetic Interaction Team

Observation of wave equations

- (M, g) Riemannian manifold
- $\triangle_g$ Laplace-Beltrami
- Ω open bounded connected subset of M
- $\omega \subset \Omega$ subset of positive measure

Wave equation

$$y_{tt} = \triangle_g y, \qquad (t,x) \in (0,T) \times \Omega,$$
$$y(0,\cdot) = y^0 \in L^2(\Omega),\ y_t(0,\cdot) = y^1 \in H^{-1}(\Omega)$$

Schrödinger equation

$$iy_t = \triangle_g y, \qquad (t,x) \in (0,T) \times \Omega,$$
$$y(0,\cdot) = y^0 \in L^2(\Omega)$$

If $\partial\Omega \neq \emptyset$, then Dirichlet boundary conditions: $y(t,\cdot)_{|\partial\Omega} = 0$.
(also: Neumann, mixed, or Robin boundary conditions)

Observable

$$z = \chi_\omega y$$

Observability

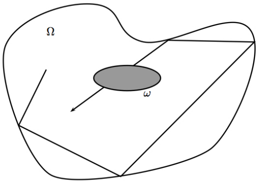
Observability inequality

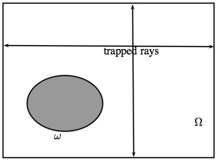
The system is said observable (in time $T > 0$) if there exists $C_T(\omega) > 0$ such that

$$\forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \qquad C_T(\omega)\|(y^0, y^1)\|^2_{L^2 \times H^{-1}} \leq \int_0^T \int_\omega y(t,x)^2 dxdt.$$

Bardos-Lebeau-Rauch (1992): the observability inequality holds if the pair (ω, T) satisfies the Geometric Control Condition (GCC) in Ω:

Every ray of geometrical optics that propagates in Ω and is reflected on its boundary $\partial\Omega$ intersects ω in time less than T.





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Question

What is the "best possible" control domain ω of fixed given measure?

Related problems

1) What is the "best domain" for achieving HUM optimal control?

$$y_{tt} - \Delta y = \chi_\omega u$$

2) What is the "best domain" domain for stabilization (with localized damping)?

$$y_{tt} - \Delta y = -k\chi_\omega y_t$$

See works by
- P. Hébrard, A. Henrot: theoretical and numerical results in 1D for optimal stabilization (for all initial data).
- A. Münch, P. Pedregal, F. Periago: numerical investigations of the optimal domain (for one fixed initial data). Study of the relaxed problem.
- S. Cox, P. Freitas, F. Fahroo, K. Ito, ...: variational formulations and numerics.
- M.I. Frecker, C.S. Kubrusly, H. Malebranche, S. Kumar, J.H. Seinfeld, ...: numerical investigations (among a finite number of possible initial data).
- K. Morris, S.L. Padula, O. Sigmund, M. Van de Wal, ...: numerical investigations for actuator placements (predefined set of possible candidates), Riccati approaches.

- ...

The model

Let $L \in (0,1)$ and $T > 0$ fixed.

It is **a priori** natural to model the problem as:

Uniform optimal design problem

Maximize

$$C_T(\omega) = \inf \left\{ \frac{\int_0^T \int_\omega y(t,x)^2 dxdt}{\|(y^0,y^1)\|^2_{L^2 \times H^{-1}}} \quad | \quad (y^0,y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \setminus \{(0,0)\} \right\}$$

or such a kind of criterion, over all possible subsets $\omega \subset \Omega$ of measure $|\omega| = L|\Omega|$.

BUT...

The model

Two difficulties arise with this model.

1. Theoretical difficulty.
2. The model is not relevant wrt practice.

The model

Two difficulties arise with this model.

1. **Theoretical difficulty.**
2. The model is not relevant wrt practice.

Spectral expansion

$-\lambda_j^2, \phi_j, \quad j \in \mathbb{N}^*$: eigenelements

Every solution can be expanded as $y(t,x) = \sum_{j=1}^{+\infty} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t))\phi_j(x)$

with $a_j = \int_\Omega y^0(x)\phi_j(x)\,dx, \quad b_j = \frac{1}{\lambda_j}\int_\Omega y^1(x)\phi_j(x)\,dx$, for every $j \in \mathbb{N}^*$. Moreover,

$\|(y^0, y^1)\|^2_{L^2 \times H^{-1}} = \sum_{j=1}^{+\infty}(a_j^2 + b_j^2)$. Then:

$$\int_0^T \int_\omega y(t,x)^2 dxdt = \int_0^T \int_\omega \left(\sum_{j=1}^{+\infty} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t))\, \phi_j(x) \right)^2 dxdt$$

$$= \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_\omega \phi_i(x)\phi_j(x)dx$$

where

$$\alpha_{ij} = \int_0^T (a_i \cos(\lambda_i t) + b_i \sin(\lambda_i t))(a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t))dt.$$

The coefficients α_{ij} depend only on the initial data (y^0, y^1).

Spectral expansion

Conclusion:

$$\int_0^T \int_\omega y(t,x)^2 dxdt = \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_\omega \phi_i(x)\phi_j(x)\, dx$$

with

$$\alpha_{ij} = \begin{cases} a_i a_j \left(\frac{\sin(\lambda_i+\lambda_j)T}{2(i+j)} + \frac{\sin(\lambda_i-\lambda_j)T}{2(\lambda_i-\lambda_j)} \right) + a_i b_j \left(\frac{1-\cos(\lambda_i+\lambda_j)T}{2(\lambda_i+\lambda_j)} - \frac{1-\cos(\lambda_i-\lambda_j)T}{2(\lambda_i-\lambda_j)} \right) \\ \quad + a_j b_i \left(\frac{1-\cos(\lambda_i+\lambda_j)T}{2(\lambda_i+\lambda_j)} + \frac{1-\cos(\lambda_i-\lambda_j)T}{2(\lambda_i-\lambda_j)} \right) + b_i b_j \left(-\frac{\sin(\lambda_i+\lambda_j)T}{2(\lambda_i+\lambda_j)} + \frac{\sin(\lambda_i-\lambda_j)T}{2(\lambda_i-\lambda_j)} \right) & \text{if } \lambda_i \neq \lambda_j, \\ a_j^2 \left(\frac{T}{2} + \frac{\sin 2\lambda_j T}{4\lambda_j} \right) + a_j b_j \left(\frac{1-\cos 2\lambda_j T}{2\lambda_j} \right) + b_j^2 \left(\frac{T}{2} - \frac{\sin 2\lambda_j T}{4\lambda_j} \right) & \text{if } \lambda_i = \lambda_j. \end{cases}$$

The coefficients α_{ij} depend only on the initial data (y^0, y^1).

Solving of the uniform design problem

$$\sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} C_T(\omega) = \sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} \inf_{\sum(a_j^2+b_j^2)=1} \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_\omega \phi_i(x)\phi_j(x)\,dx$$

$\longrightarrow$ serious difficulty due to the crossed terms.

Same difficulty in the (open) problem of determining the optimal constants in Ingham's inequalities.

The model

Two difficulties arise with this model.

1. Theoretical difficulty.
2. **The model is not relevant wrt practical expectation.**

The model

The usual observability constant is deterministic and gives an account for the worst case. It is pessimistic.

In practice: many experiments, many measures.

Objective: optimize the sensor shape and location in average.

$\longrightarrow$ randomized observability constant.

Randomized observability constant

Averaging over random initial data:

Randomized observability inequality

$$C_{T,\text{rand}}(\omega)\,\|(y^0,y^1)\|^2_{L^2\times H^{-1}} \;\leq\; \mathbb{E}\left(\int_0^T\int_\omega y_\nu(t,x)^2\,dxdt\right)$$

where $y_\nu(t,x)=\sum_{j=1}^{+\infty}\left(\beta^\nu_{1,j}a_je^{i\lambda_jt}+\beta^\nu_{2,j}b_je^{-i\lambda_jt}\right)\phi_j(x)$, with $\beta^\nu_{1,j},\beta^\nu_{2,j}$ i.i.d. Bernoulli.

(inspired from Burq-Tzvetkov, Invent. Math. 2008).

Theorem

$$C_{T,\text{rand}}(\chi_\omega)=\frac{T}{2}\inf_{j\in\mathbb{N}^*}\int_\omega\phi_j(x)^2\,dx.$$

Remark

There holds $C_{T,\text{rand}}(\chi_\omega)\geq C_T(\chi_\omega)$. There are examples where the inequality is strict.

The uniform optimal design problem

Conclusion: we model the problem as

$$\sup_{\substack{\omega\subset\Omega \\ |\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\,dx$$

Remark

This is an energy (de)concentration criterion.

Remark: another way of arriving at this criterion

Averaging in time:

Time asymptotic observability inequality:

$$C_\infty(\chi_\omega)\|(y^0,y^1)\|^2_{L^2\times H^{-1}} \leq \lim_{T\to+\infty} \frac{1}{T}\int_0^T\int_\omega |y(t,x)^2|\,dx\,dt,$$

with

$$C_\infty(\chi_\omega) = \inf\left\{ \lim_{T\to+\infty} \frac{1}{T}\frac{\int_0^T\int_\omega |y(t,x)|^2\,dx\,dt}{\|(y^0,y^1)\|^2_{L^2\times H^{-1}}} \mid (y^0,y^1)\in L^2\times H^{-1}\setminus\{(0,0)\}\right\}.$$

Theorem

If the eigenvalues of $\triangle_g$ are simple then $C_\infty(\chi_\omega) = \frac{1}{2}\inf_{j\in\mathbb{N}^*}\int_\omega \phi_j(x)^2\,dx$.

Remarks

- $C_\infty(\chi_\omega) \leq \frac{1}{2}\inf_{j\in\mathbb{N}^*}\int_\omega \phi_j(x)^2\,dx$.
- $\limsup_{T\to+\infty}\frac{C_T(\chi_\omega)}{T} \leq C_\infty(\chi_\omega)$. There are examples where the inequality is strict.

Solving of the problem

$$\sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\Omega \chi_\omega(x)\phi_j^2(x)\,dx$$

1. Convexification procedure

$$\overline{\mathcal{U}}_L = \{a \in L^\infty(\Omega,(0,1)) \mid \int_\Omega a(x)\,dx = L|\Omega|\}.$$

$$\longrightarrow \quad \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j^2(x)\,dx$$

A priori:

$$\sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j^2(x)\,dx \leq \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j^2(x)\,dx.$$

Solving of the problem

Moreover, under the assumptions

WQE (weak Quantum Ergodicity) Assumption

There exists a subsequence such that

$$\phi_j^2\, dx \rightharpoonup \frac{dx}{|\Omega|} \text{ vaguely.}$$

Uniform L^∞-boundedness

There exists $A > 0$ such that

$$\|\phi_j\|_{L^\infty} \leq A.$$

we have

$$\sup_{a \in \overline{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega a(x)\phi_j^2(x)\, dx = L$$

(reached with $a \equiv L$)

Remarks

- It is true in 1D, since $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$ on $\Omega = [0, \pi]$.
  Moreover, this relaxed problem has an infinite number of solutions, given by

$$a(x) = L + \sum_j (a_j \cos(2jx) + b_j \sin(2jx)) \text{ with } a_j \leq 0$$

(and with $|a_j|$ and $|b_j|$ small enough so that $0 \leq a(\cdot) \leq 1$).

Solving of the problem

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(reached with $a \equiv L$)

Remarks

- In multi-D:
  - L^∞-WQE is true in any flat torus.
  - If Ω is an ergodic billiard with $W^{2,\infty}$ boundary then $\phi_j^2 \, dx \rightharpoonup \frac{1}{|\Omega|} dx$ vaguely for a subset of indices of density 1.

    Gérard-Leichtnam (Duke Math. 1993), Zelditch-Zworski (CMP 1996) (see also Shnirelman, Burq-Zworski, Colin de Verdière, ...)

Solving of the problem

2. Gap or no-gap?

A priori, under WQE and uniform L^∞ boundedness assumptions:

$$\sup_{\substack{\omega\subset\Omega\\ |\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j^2(x)\,dx \quad\leq\quad \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j^2(x)\,dx \quad=\quad L.$$

Remarks in 1D:

- Note that, for every ω, $\frac{2}{\pi}\int_\omega \sin^2(jx)\,dx \to L$ as $j\to+\infty$.
- No lower semi-continuity property of the criterion.
- With $\omega_N=\bigcup_{k=1}^N \left[\frac{k\pi}{N+1}-\frac{L\pi}{2N},\frac{k\pi}{N+1}+\frac{L\pi}{2N}\right]$, one has $\chi_{\omega_N}\rightharpoonup L$ but

$$\lim_{N\to+\infty}\inf_{j\in\mathbb{N}^*}\frac{2}{\pi}\int_{\omega_N}\sin^2(jx)dx<L.$$

$\Rightarrow$ this cannot follow from usual Γ-convergence arguments.

Solving of the problem

Theorem 1

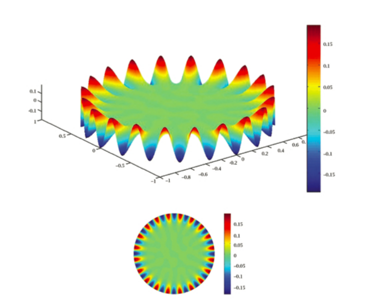
Under WQE and uniform L^∞ boundedness assumptions, there is no gap, that is:

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega \chi_\omega(x)\phi_j(x)^2\,dx = \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2\,dx = L.$$

$\rightarrow$ the maximal value of the time-asymptotic / randomized observability constant is L.

Remark

The assumptions are sufficient but not sharp:
the result also holds also true in the Euclidean disk, for which however the eigenfunctions are not uniformly bounded in L^∞ (whispering galleries phenomenon).



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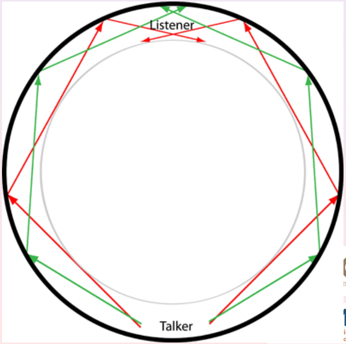


Solving of the problem

QUE (Quantum Unique Ergodicity) Assumption

We assume that $\phi_j^2\,dx \rightharpoonup \dfrac{dx}{|\Omega|}$ vaguely. (i.e. the whole sequence converges to the uniform measure)

$$\mathcal{U}_L^b = \{\chi_\omega \in \mathcal{U}_L \mid |\partial\omega| = 0\}$$

Theorem 2

Under QUE + L^p-boundedness of the ϕ_j's for some $p > 2$,

$$\sup_{\chi_\omega \in \mathcal{U}_L^b} \inf_{j \in \mathbb{N}^*} \int_\Omega \chi_\omega(x)\phi_j(x)^2\,dx = L.$$

Remark: The result holds as well if one replaces $\mathcal{U}_L^b$ with either the set of open subsets having a Lipschitz boundary, or with a bounded perimeter.

Solving of the problem

Comments on ergodicity assumptions:

- true in 1D, since $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$ on $\Omega = [0, \pi]$.
- Quantum Ergodicity property (QE) in multi-D:

  - Gérard-Leichtnam (Duke Math. 1993), Zelditch-Zworski (CMP 1996):
  If Ω is an ergodic billiard with $W^{2,\infty}$ boundary then $\phi_j^2\, dx \rightharpoonup \frac{dx}{|\Omega|}$ vaguely for a subset of indices of density 1.

  - Strictly convex billiards sufficiently regular are not ergodic (Lazutkin, 1973).
  Rational polygonal billiards are not ergodic.
  Generic polygonal billiards are ergodic (Kerckhoff-Masur-Smillie, Ann. Math. '86).

  - There exist some convex sets Ω (stadium shaped) that satisfy QE but not QUE (Hassell, Ann. Math. 2010)

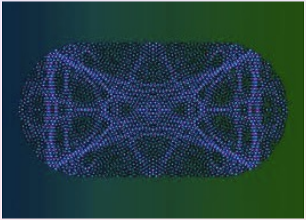
  - QUE conjecture (Rudnick-Sarnak 1994): every compact manifold having negative sectional curvature satisfies QUE.

Solving of the problem

Hence in general this assumption is related with ergodic / concentration / entropy properties of eigenfunctions.

See Shnirelman, Sarnak, Bourgain-Lindenstrauss, Colin de Verdière, Anantharaman, Nonenmacher, De Bièvre,...

If this assumption fails, we may have scars: energy concentration phenomena (there can be exceptional subsequences converging to other invariant measures, like, for instance, measures carried by closed geodesics: scars)



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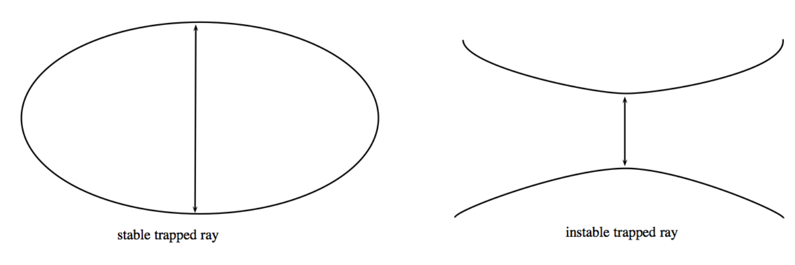

Solving of the problem

Come back to the theorem:

Under certain quantum ergodicity assumptions, there holds

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx = L.$$

Moreover:
We are able to prove that, for certain sets Ω, the second problem does not have any solution (i.e., the supremum is not reached).
We conjecture that this property is generic.

Solving of the problem

Last remark:

The proof of this no-gap result is based on a quite technical homogenization-like procedure. In dimension one, it happens that it is equivalent to the following harmonic analysis result:

Let $\mathcal{F}$ the set of functions

$$f(x) = L + \sum_{j=1}^{+\infty} (a_j \cos(2jx) + b_j \sin(2jx)), \qquad \text{with } a_j \leq 0 \quad \forall j \in \mathbb{N}^*.$$

Then:

$$d(\mathcal{F}, \mathcal{U}_L) = 0$$

but there is no $\chi_\omega \in \mathcal{F}$.

(where $\mathcal{U}_L = \{\chi_\omega \mid \omega \subset [0, \pi], \ |\omega| = L\pi\}$)

Truncated version of the second problem

Since the second problem may have no solution, it makes sense to consider as in

P. Hébrard, A. Henrot, *A spillover phenomenon in the optimal location of actuators*, SIAM J. Control Optim. **44** (2005), 349–366.

a truncated version of the second problem:

$$\sup_{\substack{\omega \subset \Omega \\ |\omega| = L|\Omega|}} \min_{1 \leq j \leq N} \int_\omega \phi_j^2(x)\, dx$$

Truncated version of the second problem

$$\sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} \min_{1\leq j\leq N} \int_\omega \phi_j^2(x)\,dx$$

Theorem

The problem has a unique solution ω^N.
Moreover, ω^N has a finite number of connected components.
If Ω has a symmetry hyperplane, then ω^N enjoys the same symmetry property.

Truncated version of the second problem

Theorem, specific to the 1D case

ω^N is symmetric with respect to $\pi/2$, is the union of at most N intervals, and:
there exists $L_N \in (0,1]$ such that, for every $L \in (0, L_N]$,

$$\int_{\omega^N} \sin^2 x\, dx = \int_{\omega^N} \sin^2(2x)\, dx = \cdots = \int_{\omega^N} \sin^2(Nx)\, dx.$$

- Equality of the criteria $\Rightarrow$ the optimal domain ω^N concentrates around the points $\frac{k\pi}{N+1}$, $k = 1, \ldots, N$.
- Spillover phenomenon: the best domain ω^N for the N rst modes is the worst possible for the $N+1$ first modes.

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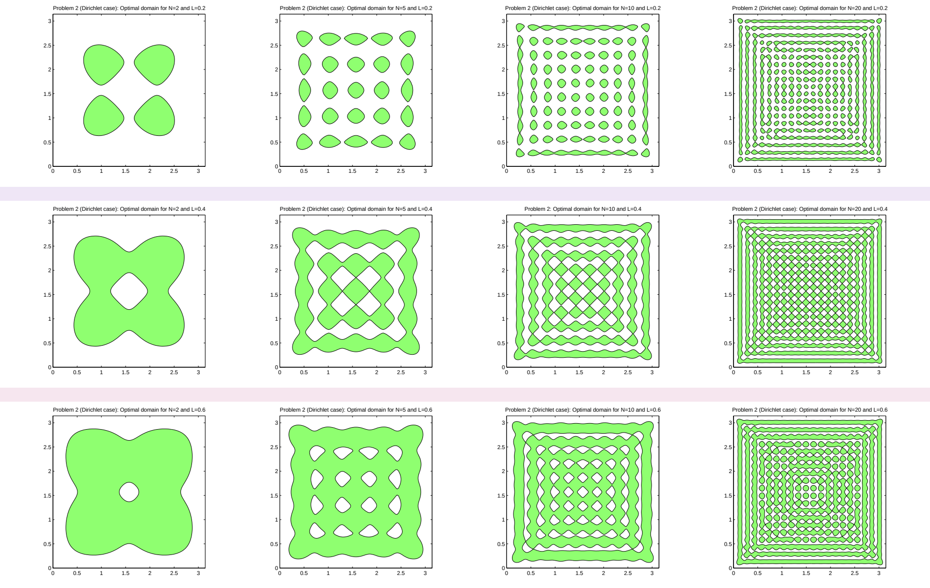
Several numerical simulations



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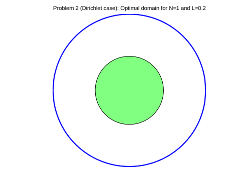
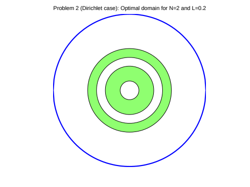
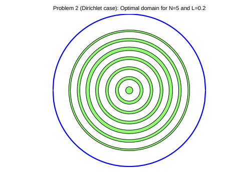
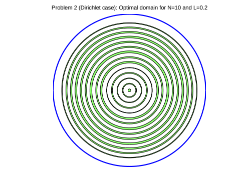
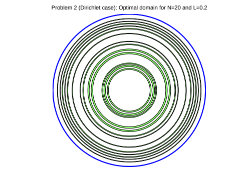










Further comments

1. Existence of a maximizer

Ensured if $\mathcal{U}_L$ is replaced with any of the following choices:

$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid P_\Omega(\omega) \leq A\} \qquad \text{(perimeter)}$$
$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid \|\chi_\omega\|_{BV(\Omega)} \leq A\} \qquad \text{(total variation)}$$
$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid \omega \text{ satisfies the } 1/A\text{-cone property}\}$$

where $A > 0$ is fixed.

Further comments

2. Intrinsic variant.

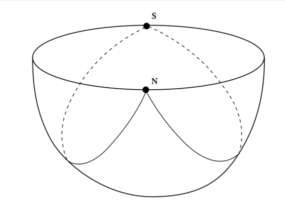
Maximize

$$\inf_{\phi\in\mathcal{E}} \int_\omega \phi(x)^2\, dx$$

over all possible subsets ω of Ω of measure $|\omega| = L|\Omega|$, where $\mathcal{E}$ denotes the set of all normalized eigenfunctions of $\triangle_g$.

$\longrightarrow$ Same no-gap results as before.

$\longrightarrow$ Examples of gap: unit sphere of $\mathbb{R}^3$, or half-unit sphere with Dirichlet boundary conditions.



Further comments

3. Parabolic equations

$$y_t = Ay$$

(for example, heat equation)

Observability inequality:

$$C_T(\chi_\omega)\|y(T,\cdot)\|_{L^2}^2 \leq \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt$$

$\longrightarrow$ The situation is very different.

Further comments

3. Parabolic equations

$$y_t = Ay$$

(for example, heat equation)

In that case, the problem is reduced (by averaging either in time or w.r.t. random initial conditions) to

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega |\phi_j(x)|^2 dx \qquad \text{with} \quad \gamma_j = \frac{e^{2\Re(\lambda_j)T} - 1}{2\Re(\lambda_j)}.$$

Theorem

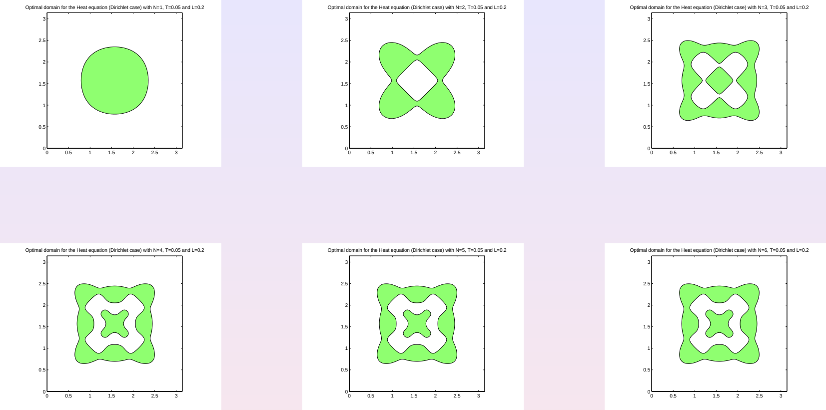
Assume that

$$\liminf_{j \to +\infty} \gamma_j(T) \int_\Omega a(x) |\phi_j(x)|^2 \, dx > \gamma_1(T),$$

for every $a \in \overline{\mathcal{U}}_L$. Then there exists $N \in \mathbb{N}^*$ such that

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega |\phi_j|^2 = \max_{\chi_\omega \in \mathcal{U}_L} \inf_{1 \le j \le n} \gamma_j \int_\omega |\phi_j|^2,$$

for every $n \geq N$. In particular there is a unique solution χ_{ω^N}. Moreover if M is analytic then ω^N is semi-analytic and has a finite number of connected components.



$L = 0.2$, $T = 0.05$, $\Omega = [0, \pi]^2$, Dirichlet boundary conditions.
$N \in \{1, 2, 3, 4, 5, 6\}$.

Conclusion and perspectives

- Same kind of analysis for the optimal design of the control domain.
- Intimate relations between domain optimization and quantum chaos (quantum ergodicity properties).

Next issues (ongoing works with Y. Privat and E. Zuazua)

- Consider other kinds of spectral criteria permitting to avoid spillover.
- Discretization issues: do the numerical optimal designs converge to the continuous optimal design as the mesh size tends to 0?

Y. Privat, E. Trélat, E. Zuazua,

- *Optimal observation of the one-dimensional wave equation*, J. Fourier Analysis Appl. (2013), to appear.
- *Optimal location of controllers for the one-dimensional wave equation*, Ann. Inst. H. Poincaré (2013), to appear.
- *Optimal observability of wave and Schrödinger equations in ergodic domains*, Preprint.
- *Optimal shape and location of sensors or controllers for parabolic equations with random initial data*, Preprint.