Moduli for Spherical Eigenmaps and Minimal Immersions

Gabor Toth
Rutgers University

Tsinghua University, Beijing

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Abstract

Eigenmaps and spherical minimal immersions of round spheres into round spheres form an interesting subject that has been studied by many authors. The components of an eigenmap are eigenfunctions of the Laplace-Beltrami operator on S^m, the Euclidean m-sphere, $m \geq 2$; and the eigenfunctions correspond to a fixed eigenvalue λ_k, $k \geq 1$. Given $m \geq 2$ and $k \geq 1$, these eigenmaps can be parametrized by a moduli space $\mathcal{L}_m^k$, a compact convex body in a finite dimensional $SO(m+1)$-module $\mathcal{E}_m^k$. Imposing conformality on eigenmaps one arrives at the concept of a spherical minimal immersion. A spherical minimal immersion is an isometric minimal immersion of S^m into a Euclidean sphere with induced metric λ_k/m-times the Euclidean metric on S^m. The corresponding moduli $\mathcal{M}_m^k$ is the intersection of $\mathcal{L}_m^k$ by an $SO(m+1)$-submodule $\mathcal{F}_m^k \subset \mathcal{E}_m^k$. The stucture of the moduli are very complex even in the lowest non-rigid cases $\mathcal{L}_m^2$ and $\mathcal{M}_m^4$, $m \geq 3$. The aim of this talk is to give a rudimentary introduction to these spaces, and the techniques for their study. Finally, we also introduce a new sequence of Minkowski type measures of symmetry for convex sets; and calculate these measures for the moduli in specific instances. This, in particular, will give a new understanding of the "roundness" of the space of minimal $SU(2)$-orbits in spheres.

1 Eigenmaps, Minimal Immersions, and Moduli

For a Euclidean vector space V, we let $S_V = \{v \in V \,|\, |v| = 1\} \subset V$ denote the unit sphere in V; in particular, $S^m = S_{\mathbf{R}^{m+1}}$ is the Euclidean m-sphere. The eigenvalues of the Laplace-Beltrami operator acting on $C^\infty(S^m)$, the real valued C^∞ functions on S^m, has eigenvalues $\lambda_k = k(k+m-1)$, $k \geq 1$. The eigenspace $\mathcal{H}_m^k \subset C^\infty(S^m)$ corresponding to the k-th eigenvalue λ_k is the space of *spherical harmonics of order* k on S^m; a spherical harmonic of order k is the restriction (to S^m) of a degree k harmonic homogeneous polynomial in $m+1$ variables $x_0, \ldots, x_m$.
A λ_k-*eigenmap* $f : S^m \to S_V$ is a map whose components $\alpha \circ f$, $\alpha \in V^*$, belong to $\mathcal{H}_m^k$. A λ_k-eigenmap $f : S^m \to S_V$ is called *full* if it has no zero component, that is, its image is not contained in any proper great sphere of S_V. Two λ_k-eigenmaps $f_1 : S^m \to S_{V_1}$ and $f_2 : S^m \to S_{V_2}$ are called *congruent* if $f_2 = U \circ f_1$ for some linear isometry $U : V_1 \to V_2$.

The archetype of a λ_k-eigenmap is the *Dirac delta map* $\delta_{\lambda_k} : S^m \to S_{(\mathcal{H}_m^k)^*}$ whose components (with respect to a scaled L^2-orthonormal basis on $\mathcal{H}_m^k \cong (\mathcal{H}_m^k)^*$) are L^2-orthonormal. (We usually fix an ortonormal basis and identify $\mathcal{H}_m^k$ with its dual.)

Remark. The concept of eigenmaps can immediately be extended to any compact Riemannian (usually assumed homogeneous) manifold as a domain. Homogeneity ensures that the Dirac delta map is a λ_k-eigenmap. In this talk we will only consider spherical domains.

Remark. A λ_k-eigenmap is harmonic in the sense of Eells-Sampson with constant energy-density $\lambda_k/2$.

Remark. For $m = 2$, the Dirac delta map is the classical Veronese map $\mathrm{Ver}_k : S^2 \to S^{2k} = S_{\mathcal{H}_2^k}$. In particular, $\mathrm{Ver}_2 : S^2 \to S^4$ factors through the antipodal map of S^2 and gives an imbedding of the real projective plane $\mathbf{R}P^2$ into S^4 with image as the classical *Veronese surface.*

Given a full λ_k-eigenmap $f : S^m \to S_V$, there is a unique surjective linear map $A : \mathcal{H}_m^k \to V$ such that $f = A \circ \delta_{\lambda_k}$. Associating to (the conguence class of) f the symmetric endomorphism $\langle f \rangle = A^\top \cdot A - I \in S^2(\mathcal{H}_m^k)$ gives rise to the DoCarmo-Wallach parametrization of the set of (congruence classes of) full λ_k-eigemaps with the compact convex body

$$\mathcal{L}_m^k = \{C \in \mathcal{E}_m^k \,|\, C + I \geq 0\},$$

of a certain linear subspace $\mathcal{E}_m^k$ of the space of traceless symmetric endomorphisms $S_0^2(\mathcal{H}_m^k) \subset S^2(\mathcal{H}_m^k)$. (Here $\geq$ stands for positive semi-definite.) $\mathcal{E}_m^k$ is defined by certain orthogonality relations in terms of the Dirac delta map. $\mathcal{L}_m^k$ is called the *moduli for* λ_k*-eigenmaps of* S^m.

A *conformal* λ_k-eigenmap $f : S^m \to S_V$ is called a *spherical minimal immersion*. The conformality factor is then λ_k/m and f is an *isometric minimal immersion* of S^m into S_V with respect to λ_k/m-times the Euclidean metric on S^m. The Dirac delta map is automatically conformal so that the DoCarmo-Wallach parametrization applies. We obtain that the set of (congruence classes of) full spherical minimal immersions with conformality factor λ_k/m can be parametrized by the compact convex body

$$\mathcal{M}_m^k = \{C \in \mathcal{F}_m^k \,|\, C + I \geq 0\},$$

where $\mathcal{F}_m^k \subset \mathcal{E}_m^k \subset S_0^2(\mathcal{H}_m^k)$ is a linear subspace defined defined by certain orthogonality relations in terms of the differential of the Dirac delta. $\mathcal{M}_m^k$ is called the *moduli for spherical minimal immersions* (with conformality λ_k/m).

Remark. The concept of spherical minimal immersions can be extended to isotropy irreducible Riemannian homogeneous domains M. Isotropy irreducibility ensures that the Dirac delta map itself is a spherical minimal immersion.

2 The $SO(m+1)$-Module Structure of $\mathcal{E}_m^k$ and $\mathcal{F}_m^k$, the Dimensions of $\mathcal{L}_m^k$ and $\mathcal{M}_m^k$

Since $SO(m+1)$ acts on S^m by isometries, the eigenspace $\mathcal{H}_m^k$ carries a natural $SO(m+1)$-module structure, and $\mathcal{E}_m^k$ and $\mathcal{F}_m^k$ are $SO(m+1)$-submodules with respect to the extended $SO(m+1)$-module structure on $S_0^2(\mathcal{H}_m^k)$. On the level of the maps, this $SO(m+1)$-action is given by precomposition so that the moduli $\mathcal{L}_m^k$ and $\mathcal{M}_m^k$ are also naturally $SO(m+1)$-invariant.

It is a classical fact that the eigenspace $\mathcal{H}_m^k$ is $SO(m+1)$-irreducible. In addition, the structure of the quotient $S_0^2(\mathcal{H}_m^k)/\mathcal{E}_m^k$, in particular, $\dim \mathcal{E}_m^k$ is known. In fact, the finite sums of products $\mathcal{H}_m^k \cdot \mathcal{H}_m^k$ of functions in $\mathcal{H}_m^k$ is an $SO(m+1)$-submodule of $S^2(\mathcal{H}_m^k)$, and

$$\mathcal{E}_m^k = S^2(\mathcal{H}_m^k)/(\mathcal{H}_m^k \cdot \mathcal{H}_m^k).$$

Moreover, we have

$$\mathcal{H}_m^k \cdot \mathcal{H}_m^k = \sum_{i=0}^{k} \mathcal{H}_{\lambda_{2i}}.$$

Combining these gives $\dim \mathcal{E}_m^k = \dim \mathcal{L}_m^k$ as

$$\dim \mathcal{L}_m^k = \binom{\binom{k+m}{m} - \binom{k+m-2}{m} + 1}{2} - \binom{2k+m}{m}.$$

It is convenient to determine the decomposition of $\mathcal{E}_m^k$ into irreducible submodules in terms of highest weights.For $0 \le l \le [k/2]$, we let $\Delta_l^k \subset \mathbf{R}^2$ be the closed convex triangle with vertices $(2l, 2l)$, (k, k) and $(2(k-l), 2l)$. We then have

$$S^2(\mathcal{H}_m^k) \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v)\in\Delta_0^k;\, u,v \text{ even}} V^{(u,v,0,\dots,0)},$$

where $V^{\mathbf{v}}$ is the complex irreducible $SO(m+1)$-module with highest weight vector $\mathbf{v} = (v_1, \dots, v_{[(m+1)/2]}) \in \mathbf{Z}^{[(m+1)/2]}$, $[(m+1)/2] = \operatorname{rank} SO(m+1)$ (with respect to the standard maximal torus providing a coordinate system for the Cartan subalgebra). (Note that, for $m = 3$, $V^{(u,v)}$ is reducible, and consists of an irreducible $SO(4)$-module and its conjugate.)
With this we have

$$\mathcal{E}_m^k \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v)\in\Delta_1^k;\, u,v \text{ even}} V^{(u,v,0,\dots,0)}.$$

This shows that the moduli space $\mathcal{L}_m^k$ parametrizing spherical λ_k-eigenmaps $f : S^m \to S_V$ is nontrival if and only if $m \ge 3$ and $k \ge 2$. (Triviality of the moduli for $m = 2$ is known as Calabi's rigidity of the Veronese maps Ver_k, $k \ge 1$.) The first nontrivial moduli $\mathcal{L}_3^2$, a 10-dimensional convex body, has been described.

The moduli $\mathcal{M}_m^k$ is much more subtle, and the decomposition of the $SO(m+1)$-module $\mathcal{F}_m^k$ into irreducible components is much more difficult. We have

$$\mathcal{F}_m^k \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v)\in\Delta_2^k;\, u,v \text{ even}} V^{(u,v,0,\dots,0)},$$

With this the exact dimension $\dim \mathcal{M}_m^k = \dim \mathcal{F}_m^k$ is known.
The moduli $\mathcal{M}_m^k$ is nontrivial if and only if $m \ge 3$ and $k \ge 4$. The first nontrivial moduli $\mathcal{M}_3^4$, an 18-dimensional convex body, has been described. In general, very little is known about the geometry of the moduli $\mathcal{L}_m^k$ and $\mathcal{M}_m^k$.

The *degree raising operator* gives rise to $SO(m+1)$-equivariant linear imbeddings $\mathcal{L}_m^k \to \mathcal{L}_m^{k+1}$ and $\mathcal{M}_m^k \to \mathcal{M}_m^{k+1}$, but the images are only *properly contained* in $\mathcal{L}_m^{k+1}$ and $\mathcal{M}_m^{k+1}$.

The *domain dimension raising operator* gives rise to linear imbeddings $\mathcal{L}_m^k \to \mathcal{L}_{m+1}^k$ and $\mathcal{M}_m^k \to \mathcal{M}_{m+1}^k$ which are equivariant with respect to the canonical homomorphism $SO(m+1) \to SO(m+2)$, and which preserve the boundaries (and the codimensions).

3 The Equivariant Moduli $(\mathcal{L}_3^k)^{SU(2)}$ and $(\mathcal{M}_3^k)^{SU(2)}$

The first nontrivial domain S^3 is special in view of the splitting of the acting group $SO(4) = SU(2) \cdot SU(2)'$. The fixed point sets $(\mathcal{L}_3^k)^{SU(2)}$ and $(\mathcal{L}_3^k)^{SU(2)'}$ are linear slices of $\mathcal{L}_3^k$. Moreover, by restriction, they are mutually orthogonal $SU(2)'$- and $SU(2)$-submodules of $\mathcal{L}_3^k$. Since they parametrize $SU(2)$- and $SU(2)'$-equivariant eigenmaps, they are called *equivariant moduli*. Note that $SU(2)'$ is a conjugate of $SU(2)$ within $SO(4)$, and the module structures on the respective equivariant moduli are isomorphic via this conjugation.

Remark. We have

$$\begin{aligned}\dim(\mathcal{L}_3^k)^{SU(2)} &= [k/2](2[k/2]+3),\\ \dim(\mathcal{M}_3^k)^{SU(2)} &= (2[k/2]+5)([k/2]-1).\end{aligned}$$

Note that both dimensions are $\mathcal{O}(k^2)$ as $k \to \infty$.

Example. The first nontrivial moduli $\mathcal{L}_3^2$ is particularly simple, as it is the convex hull of $(\mathcal{L}_3^2)^{SU(2)}$ and $(\mathcal{L}_3^2)^{SU(2)'}$. In addition, $(\mathcal{L}_3^2)^{SU(2)}$ is the convex hull of the $SU(2)'$-orbit of the parameter point $\langle Hopf \rangle$ corresponding to the Hopf map $Hopf : S^3 \to S^2$. This orbit, in turn, is the real projective plane imbedded into a copy of the 4-sphere in $(\mathcal{E}_3^2)^{SU(2)}$ as a Veronese surface. In particular, $\dim \mathcal{L}_3^2 = 2\dim(\mathcal{L}_3^2)^{SU(2)} = 10$.
In a similar vein, the 18-dimensional moduli $\mathcal{M}_3^4$ is the convex hull of the orthogonal 9-dimensional slices $(\mathcal{M}_3^4)^{SU(2)}$ and $(\mathcal{M}_3^4)^{SU(2)}$, but the structure of these slices is more subtle.

Even though much simpler than the full moduli, little is known about the $SU(2)$-equivariant moduli $(\mathcal{L}_3^k)^{SU(2)}$ and $(\mathcal{M}_3^k)^{SU(2)}$.

4 Quadratic Eigenmaps

There is a surprisingly rich variety of eigenmaps in low domain dimensions even in the quadratic case $k = 2$. By the above, we have

$$\dim \mathcal{L}_m^2 = \frac{(m-2)(m+1)(m+2)(m+3)}{12}.$$

In addition, $\mathcal{E}_m^2$ is irreducible as a $SO(m+1)$-module.

In recent years several Chinese mathematicians contributed to the understanding of quadratic eigenmaps. Many constructions (based on orthogonal multiplications using the Hopf-Whitehead construction and variants) have been obtained by Tang Zizhou.

By the example above, a full classification of quadratic eigemmaps $f : S^3 \to S_V$ can be obtained; in particular, the structure of $\mathcal{L}_3^2$ is completely known. (As a strange phenomenon, there is no full quadratic eigenmap $f : S^3 \to S^3$.)

Resolving a problem posed by J. Eells on the existence and uniqueness of self eigenmaps of spheres, Huixia He, Hui Ma, and Feng Xu showed that up to congruences on the domain and the range, there is only one quadratic eigenmap $f : S^4 \to S^4$, corresponding to the gradient of the cubic isoparametric polynomial of É. Cartan. The corresponding 35-dimensional moduli $\mathcal{L}_4^2$ is particularly interesting. As for any non-trivial orbit, the linear span of the $SO(5)$-orbit $SO(5)(\langle f \rangle)$ is the entire $\mathcal{E}_4^2$. Several natural questions arise: (1) Does the convex hull of $SO(5)(\langle f \rangle)$ give the entire $\mathcal{L}_4^2$? (2) What is the anitipodal of f? In particular, does f have L^2-orthonormal components? What is the relative moduli of the antipodal? (3) What is the maximal eigenvalue of $\langle f \rangle$, in particular, is it maximal on the boundary of $\mathcal{L}_4^2$? (4) Finally, is $SO(5)(\langle f \rangle)$ minimal in its respective S^{34}?

Most recently, Faen Wu, Yueshan Xiong, and Xinnuan Zhao gave a classification of full quadratic eigenmaps $f : S^7 \to S^7$. They showed that, up to congruences on the domain and the range, there is a singleton and a one-parameter family $f_t : S^7 \to S^7$ of such maps parametrized by the half-open interval $[0, 1)$. From their proof, it is easy to infer that on the (boundary of the) 300-dimensional moduli $\mathcal{L}_7^2$, the corresponding parameter points $\langle f_t \rangle$, $t \in [0, 1]$, fill a closed line segment with $\langle f_1 \rangle$ corresponding to the quaternionic Hop map $f_1 : S^7 \to S^4$. Once again, several questions (ranging form simple to difficult) can be asked: (1) The quadratic map f_0 has a high degree of symmetry, what is its equivariance group in $SO(8)$? What is (the dimension of) its relative moduli? The equivariance group acts on the boundary of the relative moduli; is this action transitive? (The antipodal of f_0 is a quadratic eigenmap $S^7 \to S^{26}$.) What is the dimension of the convex hull hull of the $SO(8)$-orbit of $\langle f_0 \rangle$? Etc.

5 Mean Measures of Symmetry for Convex Sets

We introduce a sequence of measures of symmetry $\{\sigma_l\}_{l\geq 1}$ for convex bodies à la Minkowski and Grünbaum. For a convex body $\mathcal{L}$ in a Euclidean vector space $\mathcal{E}$, and a point $\mathcal{O}$ in the interior of $\mathcal{L}$, $\sigma_l(\mathcal{L},\mathcal{O})$ measures how far the l-dimensional affine slices of $\mathcal{L}$ (through $\mathcal{O}$) are from being symmetric (viewed from $\mathcal{O}$). The measure of symmetry $\sigma_l(\mathcal{L},\mathcal{O})$ is defined as follows.
First, convexity of $\mathcal{L}$ implies that any line passing through $\mathcal{O}$ intersects the boundary of $\mathcal{L}$ at two *antipodal* points. If $C \in \partial\mathcal{L}$ with antipodal $C^o \in \partial\mathcal{L}$ then $\mathcal{O}$ splits the line segment $[C, C^o]$ into the ratio

$$\Lambda(C,\mathcal{O}) = \frac{d(C,\mathcal{O})}{d(C^o,\mathcal{O})},$$

where d is the distance function on $\mathcal{E}$. This defines the *distortion function* $\Lambda : \partial\mathcal{L} \to \mathbf{R}$. Clearly, $\Lambda(C^o,\mathcal{O}) = 1/\Lambda(C,\mathcal{O})$.
Second, a multi-set $\{C_0,\dots,C_l\} \subset \partial\mathcal{L}$ is called an *l-configuration* if the convex hull $[C_0,\dots,C_l]$ contains $\mathcal{O}$. The set of all l-configurations is denoted by $\mathcal{C}_l(\mathcal{L},\mathcal{O})$. We then define

$$\sigma_l(\mathcal{L},\mathcal{O}) = \inf_{\{C_0,\dots,C_l\}\in\mathcal{C}_l(\mathcal{L},\mathcal{O})} \sum_{i=0}^{l} \frac{1}{1+\Lambda(C_i,\mathcal{O})}. \tag{1}$$

Clearly, $\sigma_1(\mathcal{L},\mathcal{O}) = 1$. For $l = \dim\mathcal{L}$ the subscript is suppressed and we write $\sigma(\mathcal{L},\mathcal{O})$.

In general, we have

$$1 \leq \sigma_l(\mathcal{L},\mathcal{O}) \leq \frac{l+1}{2}, \quad l \geq 1.$$

The lower bound is attained iff $\mathcal{L}$ has an l-dimensional simplicial intersection across $\mathcal{O}$. For $l \geq 2$, the upper bound is attained iff $\mathcal{L}$ is centrally symmetric with respect to $\mathcal{O}$.

A direct consequence of Carathéodory's theorem is that the sequence $\{\sigma_l\}_{l\geq 1}$ is arithmetic form the $l = \dim\mathcal{L}$ term onwards.

Remark. The classical Minkowski measure of symmetry is

$$\mu^*(\mathcal{L}) = \inf_{\mathcal{O}\in\operatorname{int}\mathcal{L}} \sup_{C\in\partial\mathcal{L}} \Lambda(C,\mathcal{O}).$$

We have

$$\lim_{l\to\infty} \inf_{\mathcal{O}\in\operatorname{int}\mathcal{L}} \frac{\sigma_l(\mathcal{L},\mathcal{O})}{l+1} = \frac{1}{1+\mu^*(\mathcal{L})}.$$

6 Mean Measures of Symmetry of the Moduli

The Main Problem: Determine $\sigma_l(\mathcal{L}^k_m, 0)$ and $\sigma_l(\mathcal{M}^k_m, 0)$, $l \geq 2$, in particular, for $m = 3$, determine these measures for the equivariant $SU(2)$-moduli.

Remark. The distortion function $\Lambda(C, 0)$ at a boundary point $C \in \partial\mathcal{L}^k_m$ is the *maximal eigenvalue* of C as a symmetric endomorphism of $\mathcal{H}^k_m$. To determine the maximum distortion $\sup_{\partial\mathcal{L}^k_m} \Lambda(., 0)$ (and also for $\mathcal{M}^k_m$) is an important unsolved problem. (See Theorem 1 below.)

Our starting point is the following:

Theorem 1. *We have*

$$\frac{\dim \mathcal{L}^k_m + 1}{\dim \mathcal{H}^k_m} \leq \sigma(\mathcal{L}^k_m, 0) = \frac{\dim \mathcal{L}^k_m + 1}{1 + \max_{\partial\mathcal{L}^k_m} \Lambda(., 0)} \leq \frac{\dim V_{\min}}{\dim \mathcal{H}^k_m}(\dim \mathcal{L}^k_m + 1),$$

where $f : S^m \to S_{V_{\min}}$ is a λ_k-eigenmap with minimum range dimension. The same statement holds for the moduli for spherical minimal immersions. In either case above if equality holds in the upper estimate then the respective map $f : M \to S_{V_{\min}}$ has L^2-orthonormal components (up to scaling and with respect to an orthonormal basis in $V_{\min}$).

Remaks. As in the remark above, to calculate the measures of symmetry $\sigma(\mathcal{L}^k_m, 0)$ and $\sigma(\mathcal{M}^k_m, 0)$ one would need to determine the maximum distortion.
To obtain nontrivial upper bounds one needs to look for minimal ranges for which $\dim V_{\min} < \dim \mathcal{H}^k_m/2$. To determine the minimal range dimension for eigenmaps and spherical minimal immersions is the so-called *DoCarmo problem.* In general, even to give bounds on the minimum range dimension is an old and difficult problem.
Another unsolved problem (due to R.T. Smith in 1972) is to classify eigenmaps and spherical minimal immersions whose components are L^2-orthonormal.

Some arithmetic properties of the sequence $\{\sigma_l\}_{l\geq 1}$ imply:

Corollary. *Let $d^k_m = d(\mathcal{L}^k_m)$ be the maximum dimension such that $\mathcal{L}^k_m$ has a d^k_m-dimensional simplex as a linear slice (across the origin 0). Then*

$$d(\mathcal{L}^k_m) \leq \max_{\partial\mathcal{L}^k_m} \Lambda(., 0).$$

Analogous statement holds for $\mathcal{M}^k_m$ (with $\mathcal{L}^k_m$ replaced by $\mathcal{M}^k_m$). Equality holds if and only if the sequence $\{\sigma_l\}_{l\geq 1}$ is arithmetic from the d^k_m-th term onward.

Example. In the lowest non-trivial case of quadratic eigenmaps of the three-sphere, the Hopf map $Hopf : S^3 \to S^2$ corresponds to both maximal distortion 2 and minimal range dimension. Hence, we obtain

$$\sigma(\mathcal{L}_3^2, 0) = \frac{\dim \mathcal{L}_3^2 + 1}{1 + \Lambda(\langle Hopf \rangle, 0)} = 3\frac{2}{3}.$$

The explicit description of $\mathcal{L}_3^2$ shows that $\mathcal{L}_3^2$ (in fact, $(\mathcal{L}_3^2)^{SU(2)}$) has a triangular slice across 0. Thus, equality holds above, and we obtain

$$\sigma_l(\mathcal{L}_3^2, 0) = \frac{l+1}{3}, \quad l \geq 2.$$

Example. In the lowest non-trivial case of moduli $\mathcal{M}_3^4$ for quartic spherical minimal immersions of the three sphere, a role similar to the Hopf map is played by the (minimum range-dimensional) quartic minimal immersion $\mathcal{I} : S^3 \to S^9$ The corresponding point $\langle \mathcal{I} \rangle$ on the moduli has distortion 3/2 and this gives the *upper bound*

$$\sigma(\mathcal{M}_3^4, 0) \leq \frac{\dim \mathcal{M}_3^4 + 1}{1 + \Lambda(\langle \mathcal{I} \rangle, 0)} = 7\frac{3}{5}.$$

Remark. For some $SU(2)$-equivariant moduli, low dimensional simplicial slices can be constructed explicitly. For example, $(\mathcal{M}_3^6)^{SU(2)}$ has a triangular slice, and $(\mathcal{M}_3^8)^{SU(2)}$ and $(\mathcal{M}_3^{12})^{SU(2)}$ both have tetrahedral slices (across 0). These are constructed using the tetrahedral, octahedral and icosahedral spherical minimal immersions.

The *minimal orbit method* for $SU(2)$ (or *equivariant construction* originally introduced by Mashimo) has been used by DeTurck and Ziller to obtain a large number of low range-dimensional $SU(2)$-equivariant eigenmaps and spherical minimal immersions of the three sphere. They constructed these with specific invariance properties to prove that every homogeneous spherical space form (of S^3 and also of higher dimensional odd dimensional spheres) admits a minimal isometric imbedding into a Euclidean sphere (of sufficiently high dimension). For our purposes here these immersions, in turn, enable us to calculate the measures of symmetry for the equivariant moduli $(\mathcal{L}_3^k)^{SU(2)}$, $k \geq 2$, and $(\mathcal{M}_3^k)^{SU(2)}$ $k \geq 4$.

Theorem 2. *For $k \geq 2$, we have*

$$\max_{\partial(\mathcal{L}_3^k)^{SU(2)}} \Lambda(.,0) = \begin{cases} k & \text{if } k \text{ is even} \\ \frac{k-1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

The dimension $d^k = d((\mathcal{L}_3^k)^{SU(2)})$ of the largest simplicial slice of $(\mathcal{L}_3^k)^{SU(2)}$ (across 0) is equal to this maximal distortion, and we have

$$\sigma_l((\mathcal{L}_3^k)^{SU(2)}, 0) = \begin{cases} 1 & \text{if } l \leq d^k \\ \frac{l+1}{1+d^k} & \text{if } l > d^k. \end{cases}$$

In particular, we have

$$\sigma((\mathcal{L}_3^k)^{SU(2)}, 0) = \begin{cases} \frac{k+2}{2} & \text{if } k \text{ is even} \\ k & \text{if } k \text{ is odd.} \end{cases}$$

For $k \geq 5$ these hold with $\mathcal{L}_3^k$ replaced by $\mathcal{M}_3^k$, and we have

$$\sigma((\mathcal{M}_3^k)^{SU(2)}, 0) = \begin{cases} \frac{k+2}{2} - \frac{5}{k+1} & \text{if } k \text{ is even} \\ k - \frac{10}{k+1} & \text{if } k \text{ is odd.} \end{cases}$$

Since $\dim(\mathcal{L}_3^k)^{SU(2)} = \mathcal{O}(k^2)$ and $\dim(\mathcal{M}_3^k)^{SU(2)} = \mathcal{O}(k^2)$ as $k \to \infty$, these indicate that $\mathcal{L}_3^k$ and $\mathcal{M}_3^k$ are far from symmetric. Note also the interesting byproduct

$$\sigma((\mathcal{L}_3^k)^{SU(2)}, 0) > \sigma((\mathcal{M}_3^k)^{SU(2)}, 0), \quad k \geq 5$$

which is to be expected as $(\mathcal{M}_3^k)^{SU(2)}$ is a linear slice of $(\mathcal{L}_3^k)^{SU(2)}$.

Remark. For $k = 4$, the lowest range-dimensional $SU(2)$-equivariant quartic minimal immersion $\mathcal{I} : S^3 \to S^9$ gives

$$\sigma((\mathcal{M}_3^4)^{SU(2)}, 0) \leq 4.$$

Ironically, this is only an upper estimate because the $SU(2)$-module structure on the (linear) range of $\mathcal{I}$ is reducible, in fact, the double of an irreducible $SU(2)$-module. In addition, on the boundary of the moduli $(\mathcal{M}_3^4)^{SU(2)}$ there is a 6-dimensional set (corresponding to the so-called type $\mathbf{II}_0$ spherical minimal immersions). Their ranges are also reducible, the triple of an irreducible $SU(2)$-module. The corresponding parameter points are all *extremal* (in the sense of convex geometry) and their algebraic description is cumbersome.

Remark. Forgetting $SU(2)$-equivariance, the range dimensions of these $SU(2)$-equivariant eigenmaps and spherical minimal immersions can also be used for $V_{\min}$ in the upper estimate of the measures of symmetry $\sigma(\mathcal{L}_3^k, 0)$ and $\sigma(\mathcal{M}_3^k, 0)$. Only upper estimates can be expected since a least range-dimensional $SU(2)$-equivariant minimal immersion *among $SU(2)$-equivariant minimal immersions* usually do not have minimal range dimension *among all spherical minimal immersions.* This has been pointed out by Escher and Weingart who, among others, found a spherical minimal immersion $f : S^3 \to S_V$ with $k = 36$ but $\dim V \leq 36$. (For $k = 36$, the minimum range dimension for $SU(2)$-equivariant minimal immersions is 37.)

References

[1] M. Berger, Geometry I-II, Springer Verlag, 1987.

[2] A. Besse, Manifolds all of whose geodesics are closed, Springer Verlag, 1978.

[3] H. Börner, Representations of groups, North-Holland, Amsterdam, 1963.

[4] E. Calabi, Minimal immersions of surfaces in euclidean spheres, J. Diff. Geom., 1 (1967) 111-125.

[5] D. DeTurck and W. Ziller, Minimal isometric immersions of spherical space forms in spheres, J. Diff. Geom. 1 (1967) 111-125.

[6] D. DeTurck and W. Ziller, Minimal isometric immersions of spherical space forms into spheres, Comment. Math. Helvetici, 67 (1992) 428-458.

[7] D. DeTurck and W. Ziller, Spherical minimal mmersions of spherical space forms, Proc. Symp. Pure Math, Vol. 54, Part I, (1993) 111-120.

[8] M. DoCarmo and N. Wallach, Minimal immersions of spheres into spheres, Ann. Math. 93 (1971) 43-62.

[9] J. Eells and L. Lemaire, A report on harmonic maps, Bull. London Math. Soc. 10 (1978) 1-68.

[10] Ch. Escher and G. Weingart, Orbits of $SU(2)$-representations and minimal isometric immersions, Math. Ann., 316 (2000) 743-769.

[11] B. Grünbaum, Convex polytopes, Springer, 2003.

[12] B. Grünbaum, Measures of symmetry for convex sets, Proc. Sympos. Pure Math., Vol. VII (1963) 233-270.

[13] H. He, H. Ma, and F. Xu, On eigenmaps between spheres, Bull. London Math. Soc. 35 (2003) 344-354.

[14] S. Helgason, Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions, Mathematical Surveys and Monographs 83, Amer. Math. Soc. Providence, 2000.

[15] S. Helgason, Geometric analysis on symmetric spaces, Mathematical Surveys and Monographs 39, Amer. Math. Soc. Providence, 1994.

[16] S. Helgason, The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds, Acta Math. 113 (1965) 153-180.

[17] A.W. Knapp, Representation theory of semisimple groups, Princeton University Press, 1986.

[18] K. Mashimo, Minimal immersions of 3-dimensional spheres into spheres, Osaka J.Math. 2 (1984) 721 732.

[19] J.D. Moore, Isometric immersions of space forms into space forms, Pacific J. Math. 40 (1972) 157-166.

[20] G.Toth, Minimal simplices inscribed in a convex body, Geometriae Dedicata (June 2013) DOI 10.1007/s10711-013-9882-x.

[21] G.Toth, Simplicial slices of the space of minimal $SU(2)$-orbits in spheres, Contributions to Algebra and Geometry, 54 (2013) 683-699.

[22] G. Toth, On the structure of convex sets with applications to the moduli of spherical minimal immersions, Contributions to Algebra and Geometry, Vol. 49, No. 2 (2008) 491-515.

[23] G. Toth, On the shape of the moduli of spherical minimal immersions, Trans. Amer. Math. Soc. Vol. 358, No. 6 (2006) 2425-2446.

[24] G. Toth, Simplicial intersections of a convex set and moduli for spherical minimal immersions, Michigan Math. J. Vol. 52 (2004) 341-359.

[25] G. Toth, Moduli for spherical maps and spherical minimal immersions of homogeneous spaces, Journal of Lie Theory, Vol. 12 (2002) 551-570.

[26] G. Toth, Finite Möbius groups, minimal immersions of spheres, and moduli, Springer, 2002.

[27] G. Toth, Infinitesimal rotations of isometric minimal immersions between spheres, Amer. J. Math. 122 (2000) 117-152.

[28] G. Toth, Eigenmaps and the space of minimal immersions between spheres, Indiana Univ. Math. J. Vol.46, No.2 (1997) 637-658.

[29] G. Toth and W. Ziller, Spherical minimal immersions of the 3-sphere, Comment. Math. Helv. 74 (1999) 84-117.

[30] N. Wallach, Minimal immersions of symmetric spaces into spheres, in Symmetric Spaces, Dekker, New York (1972) 1-40.

[31] Weingart, G., Geometrie der Modulräume minimaler isometrischer Immersionen der Sphären in Sphären, Bonner mathematische Schriften, Nr. 314, Bonn (1999).

[32] T. Zizhou, New constructions of eigenmaps between spheres, International Journal of Mathematics, Vol. 12, No. 3 (2001) 277-288.

Department of Mathematics
Rutgers University, Camden, New Jersey 08102
e-mail: gtoth@crab.rutgers.edu

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