INVARIANT WAVE MAPS FROM GÖDEL'S UNIVERSE

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Abstract. Gödel's metric is a solution g_α to the Einstein field equations, with cosmological constant, in the presence of an incoherent matter distribution. Gödel's universe $\mathcal{G}^4_\alpha = \big(\mathbb{R}^4\,,\, g_\alpha\big)$ is the total space of a principal bundle $\mathbb{R} \to \mathcal{G}^4_\alpha \to M^3$ over a 3-dimensional nondegenerate CR manifold $M^3 = \mathcal{G}^4_\alpha / K$ got as the space of orbits of a null Killing vector field K on $\mathcal{G}^4_\alpha$. Invariant wave maps $\Phi : \mathcal{G}^4_\alpha \to N$ are precisely the vertical lifts of subelliptic harmonic maps $\phi : M^3 \to N$. For every such ϕ we solve the L^2 Dirichlet problem for the (degenerate elliptic) Jacobi operator J^ϕ_b and prove that J^ϕ_b has a discrete spectrum.

1. Why wave maps from $\mathcal{G}^4_\alpha$?

We start with a few motivational remarks[1] bringing into the picture wave maps from Gödel's universe. Gödel's metric

$$g_\alpha = -\Big(dx^0 + e^{\alpha x^1}\, dx^2\Big)^2 + (dx^1)^2 + \frac{1}{2}\, e^{2\alpha x^1} (dx^2)^2 + (dx^3)^2$$

is a solution to Einstein's field equations for an incoherent matter distribution at rest

$$R_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2}\, R(g)\, g_{\mu\nu} = \frac{8\pi\kappa}{c^2}\, T_{\mu\nu}\,,$$

$$T^{\mu\nu} = \rho\, v^\mu v^\nu\,, \quad (v^\mu) \equiv (1, 0, 0, 0),$$

$$\Lambda = -\frac{\alpha^2}{2}\,, \quad \frac{\alpha^2}{\rho} = \frac{8\pi\kappa}{c^2}\,,$$

and the field equations are the Euler-Lagrange equations of the variational principle $\delta\, S_\Omega(g) = 0$ where

$$S_\Omega(g) = \int_\Omega \left[R(g) - 2\Lambda + \frac{16\pi\kappa}{c^2}\, L \right]\, d\mathrm{v}_g$$

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where L is the Lagrangian density of matter (including all non-gravitational fields). According to Mach's principle the distribution of matter in a region of the universe should uniquely determine the geometry of that region. Yet the very same field equations above admit other solutions, such as Einsten's static solution, possessing drastically different geometric/physical properties. Consequently, Mach's principle may not be embodied into General Relativity on the ground of the field equations alone.

Brans-Dicke theory is a modification of General Relativity aiming to incorporate Mach's principle into General Relativity. A bit of heuristics is in order, to "derive" Brans-Dicke's modified action $S_\Omega(g,\,\Phi)$ from $S_\Omega(g)$. Let us look at the case where the cosmological constant is $\Lambda = 0$ and divide formally by $G = \kappa c^2$

$$\int_\Omega \left[G^{-1} R(g) + \frac{16\pi}{c^4}\, L \right]\, d\,\mathrm{v}_g\,,$$

let G vary as a function of a scalar field $\Phi : \mathbb{R}^4 \to \mathbb{R}$ i.e. take $G^{-1} = \Phi$, and add the Lagrangian density of that scalar field

$$\int_\Omega \left[\Phi\, R(g) + \frac{16\pi}{c^4}\, L - \frac{\omega}{\Phi}\, g^{\mu\nu} \Phi_{|\mu} \Phi_{|\nu} \right]\, d\,\mathrm{v}_g\,. \tag{1}$$

Cf. C. Brans and R.H. Dicke, [1]. As a further generalization of $S_\Omega(g)$ [with $\Lambda = 0$ and $L = 0$] S. Ianuş and M. Vişinescu allowed (cf. [2]) for more general values of Φ i.e. assumed that $\Phi : \mathbb{R}^4 \to N$ where N is an arbitrary Riemannian manifold, with the Riemannian metric h, and replaced the Lagrangian density of the scalar field by the trace of the bilinear form $\Phi^* h$ with respect to g. Their action reads

$$S_\Omega(g,\,\Phi) = \frac{1}{2} \int_\Omega \left[-\frac{R(g)}{2} + \frac{1}{\lambda^2}\, g^{\mu\nu} \Phi^i_{|\mu} \Phi^j_{|\nu}\, h_{ij} \circ \Phi \right]\, d\,\mathrm{v}_g$$

where λ^2 is a constant expressing the strength of the self-coupling of the scalar fields Φ^i $(1 \le i \le \dim(N))$. Note that $\Phi^{-1} g^{\mu\nu} \Phi_{|\mu} \Phi_{|\nu}$ in (1) is the trace $\mathrm{Trace}_g\,(\Phi^* h)$ with $h = t^{-1}\, dt \otimes dt$ (a Riemannian metric on $N = \mathbb{R}$). The Euler-Lagrange equations of $\delta\, S_\Omega(g,\Phi) = 0$ are

$$R_{\mu\nu} = \frac{2}{\lambda^2}\, \Phi^i_{|\mu}\, \Phi^j_{|\nu}\, h_{ij} \circ \Phi\,,$$

$$-\Box \Phi^i + \left(\Gamma^i_{jk} \circ \Phi \right)\, \Phi^j_{|\mu}\, \Phi^k_{|\nu}\, g^{\mu\nu} = 0.$$

The second set of field equations is the familiar harmonic maps system and $\Phi : \mathbb{R}^4 \to N$ is a wave map for any extremal point (g, Φ) of the action. We speculate that Brans-Dicke's theory, eventually compatible with Mach's principle, should be applied to Gödel's universe $\mathcal{G}^4_\alpha \equiv$

$(\mathbb{R}^4\, ,\, g_\alpha)$, allowing for more general values of Φ as in S. Ianuş and M. Vişinescu's work. The subject of this talk is the mathematical analysis of a particular class of wave maps $\Phi : \mathcal{G}^4_\alpha \to N$.

References

[1] C. Brans & R.H. Dicke, *Mach's principle and a relativistic theory of gravitation*, Phys. Rev., 124(1961), 925-935.

[2] S. Ianuş & M. Vişinescu, *Spontaneous compactification induced by non-linear scalar dynamics, gauge fields and submersions*, Class. Quantum Grav., 3(1986), 889-896.

2. The principal bundle $\mathbb{R} \to \mathcal{G}^4_\alpha \to M^3$

The vector field

$$K = \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^3}$$

is null [i.e. $g_\alpha(K, K) = 0$] and Killing (i.e. $\mathcal{L}_K g_\alpha = 0$). The leaf space $\mathcal{G}^4_\alpha/K$ (the space of all maximal integral curves of K) admits a C^∞ manifold structure and a nondegenerate CR structure. This may be learned from L. Koch (cf. [2]) yet appears to be known much earlier in physics (cf. I. Robinson, [3]): one may associate a natural 3-dimensional CR manifold to every Lorentzian manifold carrying a shear-free null geodesic congruence. Cf. also I. Robinson and A. Trautman, [4], who explain the phenomenon in terms of flag geometries. Originality in L. Koch's work (cf. *op. cit.*) is therefore confined to the nevertheless useful observation that the quotient space $\mathcal{G}^4_\alpha/K$ may be realized as the hyperplane $M^3 \subset \mathbb{R}^4$ of equation

$$x^0 + x^3 = 0$$

and to the explicit construction of a CR structure on M^3. This may be briefly described as follows. Straightforward integration of K shows that its maximal integral curves are the lines

$$\gamma_a(s) = s\,(e_0 - e_3) + a, \quad s \in \mathbb{R}, \quad a \in \mathbb{R}^4\, ,$$

hence

$$\mathbb{R}^4/K = \{\Gamma_a : a \in \mathbb{R}\}, \quad \Gamma_a = \gamma_a(\mathbb{R}).$$

One observes that

$$\Gamma_a = \Gamma_b \Longleftrightarrow b - a \in \mathcal{C}^1$$

where $\mathcal{C}^1$ is the line of equations

$$x^0 + x^3 = 0, \quad x^1 = x^2 = 0.$$

Let us consider the projection

$$\pi : \mathbb{R}^4 \to M^3\, , \quad \pi\left(x^0\, , x^1\, , x^2\, , x^3\right) = \left(x^0 + x^3\, , x^1\, , x^2\, , -x^0 - x^3\right),$$

mapping the line $\mathcal{C}^1$ to the origin. At this point one may identify M^3 to the quotient space $\mathbb{R}^4/\sim$ modulo the equivalence relation

$$a \sim b \Longleftrightarrow b - a \in \mathcal{C}^1$$

under the bijection

$$[a](\text{mod} \quad \sim) \longmapsto \left(a^0 + a^3\,,\, a^1\,,\, a^2\,,\, -a^0 - a^3\right) \in M^3$$

(with respect to which the projections π and $\mathbb{R}^4 \to \mathbb{R}^4/\sim$ agree).

L. Koch's observation has been completed by E. Barletta et al. (cf. [1]) with the construction of a principal bundle $\mathbb{R} \to \mathcal{G}^4_\alpha \to M^3$ whose total space is Gödel's universe. Precisely, there is a free action of $\mathbb{R}$ (the additive reals) on $\mathbb{R}^4$

$$(a, s) \in \mathbb{R}^4 \times \mathbb{R} \longmapsto a \cdot s = a + (s,\, 0,\, 0,\, -s)$$

and the synthetic object $(\mathcal{G}^4_\alpha\,,\, \pi,\, M^3\,,\, \mathbb{R})$ is a principal bundle, over M^3, with the structure group $\mathbb{R}$. The set

$$\sigma_\lambda : M^3 \to \mathbb{R}^4\,, \quad \sigma_\lambda(p) = \left(\lambda(p),\, x^1,\, x^2\,, -x^3 - \lambda(p)\right),$$

$$p = \left(-x^3\,,\, \mathbf{x}\right) \in M^3\,, \quad \mathbf{x} = \left(x^1,\, x^2,\, x^3\right) \in \mathbb{R}^3\,,$$

$$\lambda : M^3 \to \mathbb{R} \quad \text{an arbitrary function,}$$

consists of all the global sections in the principal bundle $\mathbb{R} \to \mathbb{R}^4 \to M^3$. Also $\sigma_\lambda \in C^\infty(M^3, \mathbb{R}^4)$ if and only if $\lambda \in C^\infty(M^3)$. In particular for $\lambda = 0$ one has a canonical section $\sigma_0 \in C^\infty(M^3, \mathbb{R}^4)$. Integration along the fibers is described as follows.

Theorem 1. *Let $\mathcal{J} \subset \mathbb{R}$ be a bounded open interval and $D \subset\subset M^3$ a relatively compact domain. Let*

$$\Omega = \sigma_0(D) \cdot \mathcal{J} = \{\sigma_0(p) \cdot s : p \in D,\;\; s \in \mathcal{J}\}.$$

Then $\Omega \subset \mathbb{R}^4$ is a relatively compact domain and

$$\int_\Omega (u \circ \pi)\, d\,\mathrm{v}_{g_\alpha} = \frac{|\mathcal{J}|}{\alpha\sqrt{2}} \int_D u\, \theta \wedge d\theta$$

for every continuous function $u \in C(D)$, where $|\mathcal{J}|$ is the length of $\mathcal{J}$. Also θ is the real 1-form on M^3 given by

$$\theta = dx^3 - e^{\alpha x^1}\, dx^2$$

with respect to the global chart

$$\chi : M^3 \to \mathbb{R}^3\,, \quad \chi(x^0,\, \mathbf{x}) = \mathbf{x}, \quad (x^0,\, \mathbf{x}) \in M^3.$$

References

[1] E. Barletta & S. Dragomir & M. Magliaro, *Wave maps from Gödel's universe*, Class. Quantum. Grav., 31(2014), 195001.
[2] L. Koch, *Chains on CR manifolds and Lorentz geometry*, Trans. Amer. Math. Soc., 307(1988), 827-841.
[3] I. Robinson, *Null electromagnetic fields*, J. Math. Phys., 2(1961), 290-291.
[4] I. Robinson & A. Trautman, *Cauchy-Riemann structures in optical geometry*, Proceedings of the Fourth Marcel Grossmann Meeting on General Relativity, Ed. by R. Ruffini, Elsevier Science Publishers B.V., 1986, pp. 317-324.

3. The CR structure $T_{1,0}(M^3)$

Let $Z \in C^\infty(T(M^3) \otimes \mathbb{C})$ be the complex vector field given by

$$Z = \frac{\partial}{\partial x^1} + i\sqrt{2}\left(e^{-\alpha x^1}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\right)$$

with respect to the global chart $\chi = (x^1,\, x^2,\, x^3) : M^3 \to \mathbb{R}^3$ in Theorem 1. Then (cf. L. Koch, [2])

Theorem 2.

i) *The span of Z is a CR structure $T_{1,0}(M^3)$ on M^3.*

ii) *The real differential 1-form $\theta \in \Omega^1(M^3)$ given by $\theta = dx^3 - e^{\alpha x^1}\, dx^2$ (with respect to χ) is a pseudohermitian structure on the CR manifold $(M^3, T_{1,0}(M^3))$ and the corresponding Levi form is $L_\theta(Z, \overline{Z}) = \alpha\sqrt{2}$. Consequently $T_{1,0}(M^3)$ is nondegenerate and θ is positively oriented.*

iii) *The Reeb vector of (M^3, θ) is $T = \partial/\partial x^3$.*

iv) *The maximally complex distribution $H(M^3)$ is the projection of $K^\perp \subset T(\mathcal{G}^4_\alpha)$ by $\pi : \mathbb{R}^4 \to M^3$.*

Let us consider the maps

$$\Psi : \mathbb{R}^4 \to \mathbb{C}^2\,, \quad \Psi(x) = (z, w),$$

$$z = \exp\left[\frac{\alpha}{2}\left(-x^1 + i\,\frac{1}{\sqrt{2}}\,x^3\right)\right], \quad w = -\frac{\alpha}{\sqrt{2}}\,x^2 + i\,e^{-\alpha x^1}\,,$$

$$\psi : M^3 \to \mathbb{H}_1\,, \quad \psi = f^{-1} \circ \Psi,$$

$$f : \mathbb{H}_1 \to \partial\mathcal{S}_2\,, \quad f(z,t) = (z,\, t + i\,|z|^2), \quad (z,t) \in \mathbb{H}_1\,,$$

where $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$ is the Heisenberg group and $\mathcal{S}_2 \subset \mathbb{C}^2$ is the Siegel domain. Then

v) *ψ is a local CR isomorphism of $(M^3, T_{1,0}(M^3))$ and $\mathbb{H}_1 \setminus \mathbb{R}$ with the CR structure induced by $T_{1,0}(\mathbb{H}_1)$.*

4. Wave maps

Let (N, h) be a Riemannian manifold and $\Omega \subset \mathbb{R}^4$ a relatively compact domain. Let us consider the functional

$$\mathbb{E}_\Omega : C^\infty(\mathbb{R}^4, N) \to \mathbb{R}, \quad \mathbb{E}_\Omega(\Phi) = \frac{1}{2} \int_\Omega \mathrm{trace}_g \left(\Phi^* h\right) \, d\,\mathrm{v}_g \, .$$

A *wave map* $\Phi : \mathcal{G}^4_\alpha \to N$ is a critical point $\Phi \in C^\infty(\mathbb{R}^4, N)$ of $\mathbb{E}_\Omega$ for every $\Omega \subset\subset \mathbb{R}^4$. That is

$$\frac{d}{dt} \left\{ \mathbb{E}_\Omega \left(\Phi_t\right) \right\}_{t=0} = 0$$

for any smooth 1-parameter variation $\{\Phi_t\}_{|t|<\epsilon} \subset C^\infty(\mathbb{R}^4, N)$ of Φ (i.e. $\Phi_0 = \Phi$) supported in Ω i.e. $\mathrm{Supp}(\mathbb{V}) \subset \Omega$ where $\mathbb{V} \in C^\infty(\Phi^{-1} T(N))$ is the infinitesimal variation induced by $\{\Phi_t\}_{|t|<\epsilon}$.

The first variation formula is

$$\frac{d}{dt} \left\{ \mathbb{E}_\Omega \left(\Phi_t\right) \right\}_{t=0} = - \int_\Omega h^\Phi \left(\mathbb{V}, \, \tau(\Phi)\right) \, d\, v_{g_\alpha} \, ,$$

$$\tau(\Phi)^i = -\Box \Phi^i + \left(\Gamma^i_{jk} \circ \Phi\right) \frac{\partial \Phi^j}{\partial x^\mu} \frac{\partial \Phi^k}{\partial x^\nu} \, g^{\mu\nu} \, ,$$

$$\left[g^{\mu\nu}\right] = \left[g_{\mu\nu}\right]^{-1} \, , \quad g_{\mu\nu} = g_\alpha \left(\partial_\mu \, , \, \partial_\nu\right) , \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \, ,$$

$$\Box u = \frac{\partial^2 u}{\partial (x^0)^2} - \frac{\partial^2 u}{\partial (x^1)^2} - 2 e^{-2\alpha x^1} \frac{\partial^2 u}{\partial (x^2)^2} - \frac{\partial^2 u}{\partial (x^3)^2} +$$

$$+ 4 e^{-\alpha x^1} \frac{\partial^2 u}{\partial x^0 \, \partial x^2} - \alpha \, \frac{\partial u}{\partial x^1} \, .$$

Theorem 3. *The pushforward of the wave operator $\Box$ of $\mathcal{G}^4_\alpha$ by $\pi : \mathcal{G}^4_\alpha \to M^3$ is given by*

$$\left(\pi_* \Box\right) u = \frac{\alpha}{\sqrt{2}} \, \Delta_b u, \quad u \in C^2(M^3),$$

where Δ_b is the sublaplacian of (M^3, θ) i.e.

$$\Delta_b u = -\sqrt{2} \, \frac{\partial u}{\partial x^1} -$$

$$- \frac{\sqrt{2}}{\alpha} \left\{ \frac{\partial^2 u}{\partial (x^1)^2} + 2 e^{-2\alpha x^1} \frac{\partial^2 u}{\partial (x^2)^2} + 4 e^{-\alpha x^1} \frac{\partial^2 u}{\partial x^2 \, \partial x^3} + 2 \, \frac{\partial^2 u}{\partial (x^3)^2} \right\} .$$

5. Subelliptic harmonic maps

A map $\phi \in C^\infty(M^3, N)$ is *sbelliptic harmonic* if ϕ is a critical point of the functional

$$E_D(\phi) = \frac{1}{2}\int_D \mathrm{trace}_{G_\theta}\left(\Pi_H \phi^* h\right)\, \theta \wedge d\theta$$

with respect to smooth 1-parameter variations of ϕ supported in D, for any relatively compact domain $D \subset\subset M^3$, where G_θ is the (real) Levi form of (M^3, θ) and $\Pi_H \phi^* h$ is the restriction of $\phi^* h$ to $H(M) \otimes H(M)$. The first variation formula is

$$\frac{d}{dt}\left\{E_D\left(\phi_t\right)\right\}_{t=0} = -\int_D h^\phi(V, \tau_b(\phi))\, \theta \wedge d\theta,$$

$$V = \left(\frac{\partial \phi_t}{\partial t}\right)_{t=0}, \quad \tau_b(\phi) \in C^\infty(\phi^{-1}T(N)),$$

$$\tau_b(\phi)^i = -\Delta_b \phi^i + \sum_{a=1}^{2}\left(\Gamma^i_{jk} \circ \phi\right) X_a(\phi^j)\, X_a(\phi)^k\,, \quad Z = X_1 - iX_2\,.$$

Theorem 4. *The vertical lift $\Phi = \phi \circ \pi$ of any subelliptic harmonic map ϕ of the pseudohermitian manifold (M^3, θ) into a Riemannian manifold N is a wave map. Conversely, the base map associated to any $\mathbb{R}$-invariant wave map $\Phi : \mathcal{G}^4_\alpha \to N$ is subelliptic harmonic.*

6. Degenerate elliptic Jacobi operator

Let $\mathfrak{share}(M^3, N)$ be the set of all subelliptic harmonic maps from the pseudohermitian manifold (M^3, θ) into the Riemannian manifold (N, h). Given $\phi \in \mathfrak{share}(M^3, N)$ the *rough sublaplacian* is

$$\Delta^\phi_b \mathbf{v} = -\sum_{a=1}^{2}\left\{D^\phi_{X_a} D^\phi_{X_a}\mathbf{v} - D^\phi_{\nabla_{X_a} X_a}\mathbf{v}\right\}, \quad \mathbf{v} \in C^\infty\left(\phi^{-1}T(N)\right),$$

where ∇ is the Tanaka-Webster connection of (M^3, θ). Also $D^\phi = \phi^{-1}\nabla^h$ is the pullback of ∇^h (the Levi-Civita connection of (N, h)) by ϕ (a connection in the vector bundle $\phi^{-1}T(N) \to M^3$). The symbol of Δ^ϕ_b is

$$\sigma_2\left(\Delta^\phi_b\right)_\omega v = \left[\|\omega\|^2 - g^*_{\theta,p}\left(\omega\,,\, \theta_p\right)\right] v,$$

$$\omega \in T^*_p(M^3) \setminus \{0\}, \quad v \in \left(\phi^{-1}TN\right)_p\,, \quad p \in M^3,$$

showing that Δ^ϕ_b is a degenerate elliptic operator (ellipticity degenerates in the cotangent directions θ_p, $p \in M^3$). The *subelliptic Jacobi operator* is

$$J^\phi_b : C^\infty\left(\phi^{-1}TN\right) \to C^\infty\left(\phi^{-1}TN\right),$$

$$J_b^\phi \mathbf{v} = \Delta_b^\phi \mathbf{v} - \mathrm{trace}_{G_\theta} \left[\Pi_H (R^h)^\phi (\mathbf{v},\ \phi_* \cdot) \phi_* \cdot \right],$$
$$\mathbf{v} \in C^\infty \left(\phi^{-1} TN \right).$$

J_b^ϕ differs from Δ_b^ϕ by a 0^{th} order operator, so it has the same symbol (hence it is degenerate elliptic as well). The *Hessian* of E_D at $\phi \in \mathfrak{share}(M^3, N)$ is

$$\mathrm{Hess}_b \left(E_D \right)_\phi (\mathbf{v}, \mathbf{w}) = \int_D h^\phi (J_b^\phi \mathbf{v}, \mathbf{w}) \, \theta \wedge d\theta,$$
$$D \subset\subset M^3, \quad \mathbf{v}, \mathbf{w} \in C^\infty \left(\phi^{-1} TN \right).$$

One may also introduce the *index* $\mathrm{ind}_b(\phi)$ to be the supremum of the set of dimensions $\dim_{\mathbb{R}} S$ where S ranges over the subspaces $S \subset C_0^\infty (D,\ \phi^{-1} TN)$ such that $\mathrm{Hess}_b(E_D)_\phi(\mathbf{v}, \mathbf{v}) < 0$ for any $\mathbf{v} \in S \backslash \{0\}$. A theory of stability of subelliptic harmonic maps $\phi \in \mathfrak{share}(M^3, N)$ may be developed on these lines because it can be shown that the following *second variation formula* holds

$$\frac{\partial^2}{\partial s \, \partial t} \left\{ E_D \left(\phi_{s,t} \right) \right\}_{s=t=0} = \mathrm{Hess}_b \left(E_D \right) (V, W),$$
$$\{ \phi_{s,t} \}_{-\epsilon < s,t < \epsilon} \subset C^\infty (M^3, N), \quad \phi_{0,0} = \phi,$$
$$\mathrm{Supp} \left[p \in M^3 \longmapsto (d_{(p,s,t)} f) \left(\frac{\partial}{\partial t} \right)_{(p,s,t)} \right] \subset D, \quad |t| < \epsilon, \quad |s| < \epsilon,$$
$$f : M^3 \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to N, \quad f(p,s,t) = \phi_{s,t}(p),$$
$$V = \left(\frac{\partial \phi_{s,t}}{\partial t} \right)_{s=t=0}, \quad W = \left(\frac{\partial \phi_{s,t}}{\partial t} \right)_{s=t=0}.$$

7. Spectrum of J_b^ϕ

The starting point of the stability theory for harmonic maps from a compact Riemannian manifold is to show that the Jacobi operator (of the given harmonic map) has a discrete spectrum. A subelliptic analog, holding for the subelliptic Jacobi operator J_b^ϕ of $\phi \in \mathfrak{share}(M^3, N)$, by solving the L^2 Dirichlet problem

$$J_b^\phi V = F \quad \text{in} \ \ D, \qquad V = 0 \quad \text{on} \ \ \partial D,$$

for any $F \in L^2 (D,\ \phi^{-1} TN)$. The L^2, or generalized, formulation of the Dirichlet problem above relies on the formula

$$\Delta_b^\phi = \left[\left(D^\phi \right)^H \right]^* \circ \left(D^\phi \right)^H$$

where $\left(D^{\phi}\right)^{H} V$ is the restriction of $D^{\phi}V$ to $H(M^3)$ and $\left[\left(D^{\phi}\right)^{H}\right]^{*}$ is the formal adjoint of

$$\left(D^{\phi}\right)^{H} : C^{\infty}(D, \phi^{-1}TN) \to C^{\infty}(D,\, H(M^3)^{*} \otimes \phi^{-1}TN).$$

Precisely to solve the generalized Dirichet problem is to prove the existence of $V \in \mathring{W}_{H}^{1,2}(D,\, \phi^{-1}TN)$ such that

$$\left((D^{\phi})^{H} V\,,\, (D^{\phi})^{H} S\right)_{L^2} -$$

$$-\int_D h^{\phi}\left(\operatorname{trace}_{G_\theta}\left[\Pi_H (R^h)^{\phi}(V,\, \phi_* \cdot)\phi_* \cdot\right],\, S\right)\, \theta \wedge d\theta = (F, S)_{L^2}$$

for any $S \in \mathring{W}_{H}^{1,2}(D,\, \phi^{-1}TN)$.

Theorem 5. *Let (N, h) be a Riemannian manifold of non-positive sectional curvature, such that*

$$\|R^h(A, B)C\| \le \gamma \|A\|\, \|B\|\, \|C\|, \quad A, B, C \in \mathfrak{X}(N), \tag{2}$$

for some constant $\gamma > 0$. Let $\phi \in \mathfrak{shar}(M^3, N)$ and $D \subset M^3$ a bounded domain supporting the Poincaré inequality

$$\left(\int_D \|V\|^2\, \theta \wedge d\theta\right)^{\frac{1}{2}} \le C \left(\int_D \|(D^{\phi})^H V\|^2\, \theta \wedge d\theta\right)^{\frac{1}{2}},$$

$$V \in \mathring{W}_{H}^{1,2}(D,\, \phi^{-1}TN).$$

For any $F \in L^2(D, \phi^{-1}TN)$ there is a unique generalized solution $V_F \in \mathring{W}_{H}^{1,2}(D,\, \phi^{-1}TN)$ to the L^2 Dirichlet problem for J_b^{ϕ} on D.

The estimate (2) holds for any space of constant curvature $N = N^m(k)$ with $\gamma = 2|k|$. Let us set

$$a(V, W) = \int_D \left\{ (h^{\phi})^{*}((D^{\phi})^H V\,,\, (D^{\phi})^H W) - \right.$$

$$\left. - h^{\phi}\left(\operatorname{trace}_{G_\theta}\left[\Pi_H (R^h)^{\phi}(V,\, \phi_* \cdot)\phi_* \cdot\right],\, W\right)\right\}\, \theta \wedge d\theta,$$

$$\mathcal{F}(V) = \frac{1}{2}\, a(V, W) - (F, V)_{L^2}\,, \quad V \in \mathring{W}_{H}^{1,2}(D,\, \phi^{-1}TN).$$

Poincaré inequality is crucially used in several places, and in particular to show that $\mathring{W}_{H}^{1,2}(D, \phi^{-1}TN)$ is a Hilbert space with the inner product

$$(V, W)_{\mathring{W}_{H}^{1,2}} = \left((D^{\phi})^H V\,,\, (D^{\phi})^H W\right)_{L^2}$$

(just the derivatives, rather than the inner product on $W_{H}^{1,2}(D, \phi^{-1}TN)$ where one adds the term $(V, W)_{L^2}$). The main ingredient is to show (by taking into account the curvature properties of N) that the functional $\mathcal{F}$ is strictly convex and

$$\lim_{\|V\|_{\mathring{W}_H^{1,2}}\to\infty} \mathcal{F} = +\infty.$$

As a consequence of Theorem 5 we may consider the Green operator

$$G : L^2(D,\, \phi^{-1}TN) \to L^2(D,\, \phi^{-1}TN), \quad G(F) = V_F\,,$$

whose range obeys to

$$\mathcal{R}(G) \subset \mathring{W}_H^{1,2}(D, \phi^{-1}TN).$$

The domain $D \subset M^3$ is said to satisfy the *Kondrakov condition* if the inclusion

$$\mathring{W}_H^{1,2}(D, \phi^{-1}TN) \hookrightarrow L^2(D, \phi^{-1}TN)$$

is compact. Under the assumptions of Theorem 5, for any domain $D \subset M^3$ satisfying Kondrakov's condition the Green operator G of J_b^ϕ is i) linear, ii) continuous, iii) self-adjoint, and iv) compact.

8. Generalized Dirichlet eigenvalue problem for J_b^ϕ

The Dirichlet eigenvalue problem for J_b^ϕ is

$$J_b^\phi V = \lambda V \quad \text{in } D, \qquad V = 0 \quad \text{on } \partial D.$$

To solve a weak, or generalized, version of this problem is to determine $\lambda \in \mathbb{R}$ and $V \in \mathring{W}_H^{1,2}(D, \phi^{-1}TN)$ such that

$$a(V, S) = \lambda\,(V, S)_{L^2}\,, \quad S \in \mathring{W}_H^{1,2}(D, \phi^{-1}TN).$$

Let $\sigma_{\rm gen}(J_b^\phi)$ and $\sigma(G)$ be respectively the spectrum of the generalized Dirichlet eigenvalue problem for J_b^ϕ and the spectrum of G. One may certainly relate $\sigma_{\rm gen}(J_b^\phi)$ and $\sigma(G)$ (the map is $\lambda \to 1/\lambda$) and then use i) the established properties of the Green operator and ii) standard theorems in functional analysis to get

Theorem 6. *Let $\phi \in \mathfrak{shar}(M^3, N)$. Let $D \subset M^3$ be a bounded domain supporting Poincaré's inequality and satisfying Kondrakov's compactness. Let N be a Riemannian manifold satisfying the curvature restrictions in Theorem* 5. *There is an infinite sequence*

$$0 < \lambda_1 \le \lambda_2 \le \cdots \lambda_b \le \cdots \uparrow +\infty$$

and an infinite sequence $\{V_n\}_{n\ge1} \subset \mathring{W}_H^{1,2}(D, \phi^{-1}TN)$ such that

$$\sigma_{\rm gen}(J_b^\phi) = \{\lambda_n : n \ge 1\}, \quad a(V_n, S) = \lambda_n (V_n, S)_{L^2}\,, \quad n \ge 1,$$

for any $S \in \mathring{W}_H^{1,2}(D, \phi^{-1}TN)$.