On the Large-Scale Structure of the Moduli of Eigenmaps and Spherical Minimal Immersions

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Abstract

Minimal immersions of a compact Riemannian homogeneous manifold into round spheres, or spherical minimal immersions for short, or "spherical soap bubbles," belong to a fast growing and fascinating area between algebra and geometry. This theory has rich interconnections with a variety of mathematical disciplines such as representation theory, convex geometry, harmonic maps, minimal surfaces, and orthogonal multiplications. In this survey we browse thorugh some of the developments of the theory in the last thirtysome years.

1 Eigenmaps, Minimal Immersions, and Moduli

Let M be a compact Riemannian homogeneous manifold, and λ an eigenvalue of the Laplace-Beltrami operator $\triangle$ acting on $C^\infty(M)$. Let $\mathcal{H}_\lambda \subset C^\infty(M)$ denote the eigenspace corresponding to the eigenvalue λ. A *(spherical) λ-eigenmap* $f : M \to S_V$ into the unit sphere S_V of a Euclidean vector space V is a map whose components $\alpha \circ f$, $\alpha \in V^*$, belong to $\mathcal{H}_\lambda$. A λ-eigenmap $f : M \to S_V$ is called *full* if it has no zero component, that is, its image is not contained in any proper great sphere of S_V. Two λ-eigenmaps $f_1 : M \to S_{V_1}$ and $f_2 : M \to S_{V_2}$ are called *congruent* if $f_2 = U \circ f_1$ for some linear isometry $U : V_1 \to V_2$.

Remark. λ-eigenmaps are harmonic in the sense of Eells-Sampson with constant energy-density $\lambda/2$.

The archetype of a λ-eigenmap is the *Dirac delta map* $\delta_\lambda : M \to S_{\mathcal{H}^*_\lambda}$ whose components (with respect to a scaled L^2-orthonormal basis on $\mathcal{H}_\lambda \cong \mathcal{H}^*_\lambda$) are L^2-orthonormal. (We usually fix an ortonormal basis and identify $\mathcal{H}_\lambda$ with its dual.)

Remark. For $M = S^2 = SO(3)/SO(2)$ and $\lambda_k = k(k+1)$, $k \geq 1$, the Dirac delta map is the classical Veronese maps $\mathrm{Ver}_k : S^2 \to S^{2k} = S_{\mathcal{H}_{\lambda_k}}$. In particular, $\mathrm{Ver}_2 : S^2 \to S^4$ factors through the antipodal map of S^2 and gives an imbedding of the real projective plane $\mathbf{R}P^2$ into S^4 with image as the classical *Veronese surface.*

Given a full λ-eigenmap $f : M \to S_V$, there is a unique surjective linear map $A : \mathcal{H}_\lambda \to V$ such that $f = A \circ \delta_\lambda$. Associating to (the conguence class of) f the symmetric endomorphism $\langle f \rangle = A^\top \cdot A - I \in S^2(\mathcal{H}_\lambda)$ gives rise to the DoCarmo-Wallach parametrization of the set of (congruence classes of) full λ-eigemaps with the compact convex body

$$\mathcal{L}_\lambda = \{C \in \mathcal{E}_\lambda \,|\, C + I \geq 0\},$$

of a certain linear subspace $\mathcal{E}_\lambda$ of the space of traceless symmetric endomorphisms $S^2_0(\mathcal{H}_\lambda) \subset S^2(\mathcal{H}_\lambda)$. (Here $\geq$ stands for positive semi-definite.) $\mathcal{E}_\lambda$ is defined by certain orthogonality relations in terms of the Dirac delta map. $\mathcal{L}_\lambda$ is called the *moduli for λ-eigenmaps.*

Assuming that M is *isotropy irreducible*, a *conformal* λ-eigenmap $f : M \to S_V$ is called a *spherical minimal immersion.* The conformality factor is then $\lambda/\dim M$ and f is an *isometric minimal immersion* of M into S_V with respect to $\lambda/\dim M$-times the original metric on M. Due to isotropy irreducibility, the Dirac delta map is automatically conformal so that the DoCarmo-Wallach parametrization applies.

We obtain that the set of (congruence classes of) spherical minimal immersions with conformality factor $\lambda/\dim M$ can be parametrized by the compact convex body

$$\mathcal{M}_\lambda = \{C \in \mathcal{F}_\lambda \,|\, C + I \geq 0\},$$

where $\mathcal{F}_\lambda \subset \mathcal{E}_\lambda \subset S_0^2(\mathcal{H}_\lambda)$ is a linear subspace defined defined by certain orthogonality relations in terms of the differential of the Dirac delta. $\mathcal{M}_\lambda$ is called the *moduli for spherical minimal immersions* (with conformality $\lambda/\dim M$).

Remark. Beyond the fact that the moduli $\mathcal{L}_\lambda$ and $\mathcal{M}_\lambda$ are convex bodies in their ambient linear spans $\mathcal{E}_\lambda$ and $\mathcal{F}_\lambda$, very little are known about their structures.

2 The G-Module Structure of $\mathcal{E}_\lambda$ and $\mathcal{F}_\lambda$, the Dimensions of $\mathcal{E}_\lambda$ and $\mathcal{M}_\lambda$

If G is a transitive Lie group of isometries of M, then the eigenspace $\mathcal{H}_\lambda$ carries a natural G-module structure, and $\mathcal{E}_\lambda$ and $\mathcal{F}_\lambda$ are G-submodules with respect to the extended G-module structure on $S_0^2(\mathcal{H}_\lambda)$. On the level of the spherical maps, this G-action is given by precomposition so that the moduli $\mathcal{L}_\lambda$ and $\mathcal{M}_\lambda$ are also naturally G-invariant.

For a compact rank one symmetric space $M = G/K$ $(K \subset G)$, the eigenspaces $\mathcal{H}_\lambda$ are irreducible, and the structure of the quotient $S_0^2(\mathcal{H}_\lambda)/\mathcal{E}_\lambda$, in particular, $\dim \mathcal{E}_\lambda$ is known. In fact, the finite sums of products $\mathcal{H}_\lambda \cdot \mathcal{H}_\lambda$ of functions in $\mathcal{H}_\lambda$ is a G-submodule of $S^2(\mathcal{H}_\lambda)$, and

$$\mathcal{E}_\lambda = S^2(\mathcal{H}_\lambda)/(\mathcal{H}_\lambda \cdot \mathcal{H}_\lambda).$$

If $\{\lambda_k\}_{k\geq 1}$ denotes the sequence of eigenvalues in increasing order, then we have

$$\mathcal{H}_{\lambda_k} \cdot \mathcal{H}_{\lambda_k} = \begin{cases} \sum_{i=0}^{k} \mathcal{H}_{\lambda_{2i}} & \text{if } M = S^m \\ \sum_{i=0}^{2k} \mathcal{H}_{\lambda_i} & \text{otherwise} \end{cases}$$

Combining these gives $\dim \mathcal{E}_\lambda = \dim \mathcal{L}_\lambda$.

For the Euclidean sphere $M = S^m$ and $G = SO(m+1)$, we write $\mathcal{H}_m^k = \mathcal{H}_{\lambda_k}$, $\mathcal{E}_m^k = \mathcal{E}_{\lambda_k}$, etc. The decomposition of $\mathcal{E}_m^k$ into irreducible $SO(m+1)$-components (in terms of highest weights) has been determined. For $0 \leq l \leq [k/2]$, we let $\Delta_l^k \subset \mathbf{R}^2$ be the closed convex triangle with vertices $(2l, 2l)$, (k, k) and $(2(k-l), 2l)$. We then have

$$S^2(\mathcal{H}_m^k) \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v)\in\Delta_0^k;\, u,v \text{ even}} V^{(u,v,0,\dots,0)},$$

where $V^{\mathbf{v}}$ is the complex irreducible $SO(m+1)$-module with highest weight vector $\mathbf{v} = (v_1, \ldots, v_{[(m+1)/2]}) \in \mathbf{Z}^{[(m+1)/2]}$, $[(m+1)/2] = \operatorname{rank} SO(m+1)$ (with respect to the standard maximal torus providing a coordinate system for the Cartan subalgebra). With this we have

$$\mathcal{E}_m^k \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v) \in \Delta_1^k;\, u,v \text{ even}} V^{(u,v,0,\ldots,0)}.$$

This shows that the moduli space $\mathcal{L}_m^k$ parametrizing spherical λ_k-eigenmaps $f : S^m \to S_V$ is nontrival if and only if $m \geq 3$ and $k \geq 2$. (Triviality of the moduli for $m = 2$ is known as Calabi's rigidity of the Veronese maps Ver_k, $k \geq 1$.) The first nontrivial moduli $\mathcal{L}_3^2$, a 10-dimensional convex body, has been described. (See the next section.)

The moduli $\mathcal{M}_\lambda$ has been extensively studied only for the Euclidean m-sphere S^m and $G = SO(m+1)$. (This is partially due to the complexity of the decomposition of $\mathcal{F}_\lambda$ into irreducible components for non-spherical compact rank one symmetric spaces. For example, for the complex projective space, $\mathcal{F}_\lambda$ fails to have multiplicity one decomposition.) For $M = S^m$, we have

$$\mathcal{F}_m^k \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v) \in \Delta_2^k;\, u,v \text{ even}} V^{(u,v,0,\ldots,0)},$$

With this the decomposition of $\mathcal{F}_m^k$ into irreducible $SO(m+1)$-components is determined, in particular, the exact dimension $\dim \mathcal{M}_m^k = \dim \mathcal{F}_m^k$ is known.
The moduli $\mathcal{M}_m^k$ is nontrivial if and only if $m \geq 3$ and $k \geq 4$. The first nontrivial moduli $\mathcal{M}_3^4$, an 18-dimensional convex body, has been described. (See the next section.) In general, very little is known about the geometry of the moduli $\mathcal{L}_m^k$ and $\mathcal{M}_m^k$.

The degree raising operator gives rise to $SO(m+1)$-equivariant linear imbeddings $\mathcal{L}_m^k \to \mathcal{L}_m^{k+1}$ and $\mathcal{M}_m^k \to \mathcal{M}_m^{k+1}$, but the images are only *properly contained* in linear slices of $\mathcal{L}_m^{k+1}$ and $\mathcal{M}_m^{k+1}$.

3 The Equivariant Moduli $(\mathcal{L}_3^k)^{SU(2)}$ and $(\mathcal{M}_3^k)^{SU(2)}$

The first nontrivial domain S^3 is special in view of the splitting of the acting group $SO(4) = SU(2) \cdot SU(2)'$. The fixed point sets $(\mathcal{L}_3^k)^{SU(2)}$ and $(\mathcal{L}_3^k)^{SU(2)'}$ are linear slices of $\mathcal{L}_3^k$. Moreover, by restriction, they are mutually orthogonal $SU(2)'$- and $SU(2)$-submodules of $\mathcal{L}_3^k$. Since they parametrize $SU(2)$- and $SU(2)'$-equivariant eigenmaps, they are called *equivariant moduli.* Note that $SU(2)'$ is a conjugate of $SU(2)$ within

$SO(4)$, and the module structures on the respective equivariant moduli are isomorphic via this conjugation.

Remark. We have

$$
\begin{aligned}
\dim(\mathcal{L}_3^k)^{SU(2)} &= [k/2](2[k/2]+3),\\
\dim(\mathcal{M}_3^k)^{SU(2)} &= (2[k/2]+5)([k/2]-1).
\end{aligned}
$$

Note that both dimensions are $\mathcal{O}(k^2)$ as $k \to \infty$.

Example. The first nontrivial moduli $\mathcal{L}_3^2$ is particularly simple, as it is the convex hull of $(\mathcal{L}_3^2)^{SU(2)}$ and $(\mathcal{L}_3^2)^{SU(2)'}$. In addition, $(\mathcal{L}_3^2)^{SU(2)}$ is the convex hull of the $SU(2)'$-orbit of the parameter point $\langle Hopf \rangle$ corresponding to the Hopf map $Hopf : S^3 \to S^2$. This orbit, in turn, is the real projective plane imbedded into a copy of the 4-sphere in $(\mathcal{E}_3^2)^{SU(2)}$ as a Veronese surface. In particular, $\dim \mathcal{L}_3^2 = 2\dim(\mathcal{L}_3^2)^{SU(2)} = 10$.
In a similar vein, $\mathcal{M}_3^4$ is the convex hull of the orthogonal 9-dimensional slices $(\mathcal{M}_3^4)^{SU(2)}$ and $(\mathcal{M}_3^4)^{SU(2)}$, but the structure of these slices is more subtle.

Even though much simpler than the full moduli, little is known about the $SU(2)$-equivariant moduli $(\mathcal{L}_3^k)^{SU(2)}$ and $(\mathcal{M}_3^k)^{SU(2)}$.

Examples. There is a surprisingly rich variety of eigenmaps in low domain dimensions even in the quadratic case $k = 2$. Many constructions (based on orthogonal multiplications using the Hopf-Whitehead construction and variants) have been obtained by T. Zizhou. Resolving a problem posed by J. Eells on the existence and uniqueness of self eigenmaps of spheres, H. He, H. Ma, and F. Xu showed that up to congruence there is only one quadratic eigenmap $f : S^4 \to S^4$. Most recently, F. Wu, Y. Xiong, and X. Zhao gave a full classification of quadratic eigenmaps $f : S^7 \to S^7$.

4 Moduli for Harmonic Non-Holomorphic Polynomial Maps Between Complex Projective Spaces

Let $\mathcal{H}_m^{p,q}$, $p, q \geq 0$, denote the complex vector space of harmonic homogeneous polynomials on $\mathbf{C}^{m+1}$ of bidegree (p, q) (degree p in the complex variables $z_0, \ldots, z_m$ and degree q in the conjugates $\bar{z}_0, \ldots, \bar{z}_m$). Then $\mathcal{H}_m^{p,q}$ is an irreducible $U(m+1)$-module; in fact, it is a $U(m+1)$-component of the restriction $\mathcal{H}_{2m+1}^{p+q} \otimes_{\mathbf{R}} \mathbf{C}|_{U(m+1)}$ (with $S^{2m+1} \subset \mathbf{C}^{m+1} = \mathbf{R}^{2(m+1)}$ being the unit sphere).

A λ_{p+q}-eigenmap $f : S^{2m+1} \to S^{2n+1}$ is said to be a (complex) *eigenmap of bidegree* (p, q) if the components of f belong to $\mathcal{H}_m^{p,q}$. An eigenmap f of bidegree (p, q) factors through the canonical projections $S^{2m+1} \to \mathbf{C}P^m$ and $S^{2n+1} \to \mathbf{C}P^n$ giving a harmonic map $\tilde{f} : \mathbf{C}P^m \to \mathbf{C}P^n$ (which is non-(anti)holomorphic if $p, q > 0$). The DoCarmo-Wallach moduli space construction can be adapted to this unitary setting giving moduli spaces of such eigenmaps. Representation theory of the unitary group (Littlewood-Richardson type multiplicity formulas due to D. Barbasch) give lower bounds on the corresponding moduli.

5 Mean Measures of Symmetry for Convex Sets

We introduce a sequence of measures of symmetry $\{\sigma_l\}_{l\geq 1}$ for convex bodies à la Minkowski and Grünbaum. For a convex body $\mathcal{L}$ in a Euclidean vector space $\mathcal{E}$, and a point $\mathcal{O}$ in the interior of $\mathcal{L}$, $\sigma_l(\mathcal{L}, \mathcal{O})$ measures how far the l-dimensional affine slices of $\mathcal{L}$ (through $\mathcal{O}$) are from being symmetric (viewed from $\mathcal{O}$). The measure of symmetry $\sigma_l(\mathcal{L}, \mathcal{O})$ is defined as follows.

First, convexity of $\mathcal{L}$ implies that any line passing through $\mathcal{O}$ intersects the boundary of $\mathcal{L}$ at two *antipodal* points. If $C \in \partial\mathcal{L}$ with antipodal $C^o \in \partial\mathcal{L}$ then $\mathcal{O}$ splits the line segment $[C, C^o]$ into the ratio

$$\Lambda(C, \mathcal{O}) = \frac{d(C, \mathcal{O})}{d(C^o, \mathcal{O})},$$

where d is the distance function on $\mathcal{E}$. This defines the *distortion function* $\Lambda : \partial\mathcal{L} \to \mathbf{R}$. Clearly, $\Lambda(C^o, \mathcal{O}) = 1/\Lambda(C, \mathcal{O})$.

Second, a multi-set $\{C_0, \ldots, C_l\} \subset \partial\mathcal{L}$ is called an *l-configuration* if the convex hull $[C_0, \ldots, C_l]$ contains $\mathcal{O}$. The set of all l-configurations is denoted by $\mathcal{C}_l(\mathcal{L}, \mathcal{O})$. We then define

$$\sigma_l(\mathcal{L}, \mathcal{O}) = \inf_{\{C_0,\ldots,C_l\}\in\mathcal{C}_l(\mathcal{L},\mathcal{O})} \sum_{i=0}^{l} \frac{1}{1 + \Lambda(C_i, \mathcal{O})}. \tag{1}$$

Clearly, $\sigma_1(\mathcal{L}, \mathcal{O}) = 1$. For $l = \dim \mathcal{L}$ the subscript is suppressed and we write $\sigma(\mathcal{L}, \mathcal{O})$.

In general, we have

$$1 \leq \sigma_l(\mathcal{L}, \mathcal{O}) \leq \frac{l+1}{2}, \quad l \geq 1.$$

The lower bound is attained iff $\mathcal{L}$ has an l-dimensional simplicial intersection across $\mathcal{O}$. For $l \geq 2$, the upper bound is attained iff $\mathcal{L}$ is centrally symmetric with respect to $\mathcal{O}$.

A direct consequence of Carathéodory's theorem is that the sequence $\{\sigma_l\}_{l\geq 1}$ is arithmetic form the $l = \dim \mathcal{L}$ term onwards.

Remark. The classical Minkowski measure of symmetry is

$$\mu^*(\mathcal{L}) = \inf_{\mathcal{O} \in \operatorname{int} \mathcal{L}} \sup_{C \in \partial \mathcal{L}} \Lambda(C, \mathcal{O}).$$

We have

$$\lim_{l \to \infty} \inf_{\mathcal{O} \in \operatorname{int} \mathcal{L}} \frac{\sigma_l(\mathcal{L}, \mathcal{O})}{l+1} = \frac{1}{1 + \mu^*(\mathcal{L})}.$$

6 Mean Measures of Symmetry of the Moduli

The Main Problem: Determine $\sigma_l(\mathcal{L}_\lambda, 0)$ and $\sigma_l(\mathcal{M}_\lambda, 0)$, $l \geq 2$, in particular, determine these measures for the moduli for the sphere $M = S^m$ and, for $m = 3$, for the equivariant $SU(2)$-moduli.

Remark. The distortion function $\Lambda(C, 0)$ at a boundary point $C \in \partial \mathcal{L}_\lambda$ is the *maximal eigenvalue* of C as a symmetric endomorphism of $\mathcal{H}_\lambda$. To determine the maximum distortion $\sup_{\partial \mathcal{L}_\lambda} \Lambda(., 0)$ (and also for $\mathcal{M}_\lambda$) is an important unsolved problem. (See Theorem 1 below.)

Our starting point is the following:

Theorem 1. *Let M be a Riemannian homogeneous space. Assume that the eigenspace $\mathcal{H}_\lambda$ is an irreducible G-submodule. Then, we have*

$$\frac{\dim \mathcal{L}_\lambda + 1}{\dim \mathcal{H}_\lambda} \leq \sigma(\mathcal{L}_\lambda, 0) = \frac{\dim \mathcal{L}_\lambda + 1}{1 + \max_{\partial \mathcal{L}_\lambda} \Lambda(., 0)} \leq \frac{\dim V_{\min}}{\dim \mathcal{H}_\lambda} (\dim \mathcal{L}_\lambda + 1),$$

where $f : M \to S_{V_{\min}}$ is a spherical λ-eigenmap with minimum range dimension. If M is isotropy irreducible then we have

$$\frac{\dim \mathcal{M}_\lambda + 1}{\dim \mathcal{H}_\lambda} \leq \sigma(\mathcal{M}_\lambda, 0) = \frac{\dim \mathcal{M}_\lambda + 1}{1 + \max_{\partial \mathcal{M}_\lambda} \Lambda(., 0)} \leq \frac{\dim V_{\min}}{\dim \mathcal{H}_\lambda} (\dim \mathcal{M}_\lambda + 1),$$

where $f : M \to S_{V_{\min}}$ is a spherical minimal immersion (inducing $\lambda / \dim M$ times the metric on M) with minimum range dimension. In either case above if equality holds in the upper estimate then the respective map $f : M \to S_{V_{\min}}$ has L^2-orthonormal components (up to scaling and with respect to an orthonormal basis in $V_{\min}$).

Remaks. As noted above, for compact rank one symmetric spaces M the eigenspaces $\mathcal{H}_\lambda$ are irreducible so that Theorem 1 applies. As in the remark above, to calculate the measures of symmetry $\sigma(\mathcal{L}_\lambda, 0)$ and $\sigma(\mathcal{M}_\lambda, 0)$ one would need to determine the maximum distortion.
To obtain nontrivial upper bounds one needs to look for minimal ranges for which $\dim V_{\min} < \dim \mathcal{H}_\lambda/2$. To determine the minimal range dimension for eigenmaps and spherical minimal immersions is the so-called *DoCarmo problem.* In general, even to give bounds on the minimum range dimension is an old and difficult problem.
Another unsolved problem (due to R.T. Smith in 1972) is to classify eigenmaps and spherical minimal immersions whose components are L^2-orthonormal.

Some arithmetic properties of the sequence $\{\sigma_l\}_{l\geq 1}$ imply:

Corollary. *Let $d_\lambda = d(\mathcal{L}_\lambda)$ be the maximum dimension such that $\mathcal{L}_\lambda$ has a d_λ-dimensional simplex as a linear slice (across the origin 0). Then*

$$d(\mathcal{L}_\lambda) \leq \max_{\partial \mathcal{L}_\lambda} \Lambda(.,0).$$

Analogous statement holds for $\mathcal{M}_\lambda$ (with $\mathcal{L}_\lambda$ replaced by $\mathcal{M}_\lambda$). Equality holds if and only if the sequence $\{\sigma_l\}_{l\geq 1}$ is arithmetic from the d_λ-th term onward.

Example. In the lowest non-trivial case of quadratic eigenmaps of the three-sphere, the Hopf map *Hopf* $: S^3 \to S^2$ corresponds to both maximal distortion 2 and minimal range dimension. Hence, we obtain

$$\sigma(\mathcal{L}_3^2, 0) = \frac{\dim \mathcal{L}_3^2 + 1}{1 + \Lambda(\langle \mathit{Hopf} \rangle, 0)} = 3\frac{2}{3}.$$

The explicit description of $\mathcal{L}_3^2$ shows that $\mathcal{L}_3^2$ (in fact, $(\mathcal{L}_3^2)^{SU(2)}$) has a triangular slice across 0. Thus, equality holds above, and we obtain

$$\sigma_l(\mathcal{L}_3^2, 0) = \frac{l+1}{3}, \quad l \geq 2.$$

Example. In the lowest non-trivial case of moduli $\mathcal{M}_3^4$ for quartic spherical minimal immersions of the three sphere, a role similar to the Hopf map is played by the (minimum range-dimensional) quartic minimal immersion $\mathcal{I} : S^3 \to S^9$ The corresponding point $\langle \mathcal{I} \rangle$ on the moduli has distortion $3/2$ and this gives the *upper bound*

$$\sigma(\mathcal{M}_3^4, 0) \leq \frac{\dim \mathcal{M}_3^4 + 1}{1 + \Lambda(\langle \mathcal{I} \rangle, 0)} = 7\frac{3}{5}.$$

Remark. For some $SU(2)$-equivariant moduli, low dimensional simplicial slices can be constructed explicitly. For example, $(\mathcal{M}_3^6)^{SU(2)}$ has a triangular slice, and $(\mathcal{M}_3^8)^{SU(2)}$ and $(\mathcal{M}_3^{12})^{SU(2)}$ both have tetrahedral slices (across 0). These are constructed using the tetrahedral, octahedral and icosahedral spherical minimal immersions.

The *minimal orbit method* for $SU(2)$ (or *equivariant construction* originally introduced by Mashimo) has been used by DeTurck and Ziller to obtain a large number of low range-dimensional $SU(2)$-equivariant eigenmaps and spherical minimal immersions of the three sphere. They constructed these with specific invariance properties to prove that every homogeneous spherical space form (of S^3 and also of higher dimensional odd dimensional spheres) admits a minimal isometric imbedding into a Euclidean sphere (of sufficiently high dimension). For our purposes here these immersions, in turn, enable us to calculate the measures of symmetry for the equivariant moduli $(\mathcal{L}_3^k)^{SU(2)}$, $k \geq 2$, and $(\mathcal{M}_3^k)^{SU(2)}$ $k \geq 4$.

Theorem 2. *For $k \geq 2$, we have*

$$\max_{\partial(\mathcal{L}_3^k)^{SU(2)}} \Lambda(.,0) = \begin{cases} k & \textit{if } k \textit{ is even} \\ \frac{k-1}{2} & \textit{if } k \textit{ is odd.} \end{cases}$$

The dimension $d^k = d((\mathcal{L}_3^k)^{SU(2)})$ of the largest simplicial slice of $(\mathcal{L}_3^k)^{SU(2)}$ (across 0) is equal to this maximal distortion, and we have

$$\sigma_l((\mathcal{L}_3^k)^{SU(2)},0) = \begin{cases} 1 & \textit{if } l \leq d^k \\ \frac{l+1}{1+d^k} & \textit{if } l > d^k. \end{cases}$$

In particular, we have

$$\sigma((\mathcal{L}_3^k)^{SU(2)},0) = \begin{cases} \frac{k+2}{2} & \textit{if } k \textit{ is even} \\ k & \textit{if } k \textit{ is odd.} \end{cases}$$

For $k \geq 5$ these hold with $\mathcal{L}_3^k$ replaced by $\mathcal{M}_3^k$, and we have

$$\sigma((\mathcal{M}_3^k)^{SU(2)},0) = \begin{cases} \frac{k+2}{2} - \frac{5}{k+1} & \textit{if } k \textit{ is even} \\ k - \frac{10}{k+1} & \textit{if } k \textit{ is odd.} \end{cases}$$

Since $\dim(\mathcal{L}_3^k)^{SU(2)} = \mathcal{O}(k^2)$ and $\dim(\mathcal{M}_3^k)^{SU(2)} = \mathcal{O}(k^2)$ as $k \to \infty$, these indicate that $\mathcal{L}_3^k$ and $\mathcal{M}_3^k$ are far from symmetric. Note also the interesting byproduct

$$\sigma((\mathcal{L}_3^k)^{SU(2)},0) > \sigma((\mathcal{M}_3^k)^{SU(2)},0), \quad k \geq 5$$

which is to be expected as $(\mathcal{M}_3^k)^{SU(2)}$ is a linear slice of $(\mathcal{L}_3^k)^{SU(2)}$.

Remark. For $k = 4$, the lowest range-dimensional $SU(2)$-equivariant quartic minimal immersion $\mathcal{I} : S^3 \to S^9$ gives

$$\sigma((\mathcal{M}_3^4)^{SU(2)}, 0) \leq 4.$$

Ironically, this is only an upper estimate because the $SU(2)$-module structure on the (linear) range of $\mathcal{I}$ is reducible, in fact, the double of an irreducible $SU(2)$-module. In addition, on the boundary of the moduli $(\mathcal{M}_3^4)^{SU(2)}$ there is a 6-dimensional set (corresponding to the so-called type $\mathbf{II}_0$ spherical minimal immersions.) Their ranges are also reducible, the triple of an irreducible $SU(2)$-module. The corresponding parameter points are all *extremal* (in the sense of convex geometry) and their algebraic description is cumbersome.

Remark. Forgetting $SU(2)$-equivariance, the range dimensions of these $SU(2)$-equivariant eigenmaps and spherical minimal immersions can also be used for $V_{\min}$ in the upper estimate of the measures of symmetry $\sigma(\mathcal{L}_3^k, 0)$ and $\sigma(\mathcal{M}_3^k, 0)$. Only upper estimates can be expected since a least range-dimensional $SU(2)$-equivariant minimal immersion *among* $SU(2)$*-equivariant minimal immersions* usually do not have minimal range dimension *among all spherical minimal immersions.* This has been pointed out by Escher and Weingart who, among others, found a spherical minimal immersion $f : S^3 \to S_V$ with $k = 36$ but $\dim V \leq 36$. (For $k = 36$, the minimum range dimension for $SU(2)$-equivariant minimal immersions is 37.)

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