# On $SL(2,\mathbb{R})$ valued cocycles of Hölder class with zero exponent over Kronecker flows

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**Abstract**

We show that a generic $SL(2,\mathbb{R})$ valued cocycle in the class of $C^r$, $(0 < r < 1)$ cocycles based on a rotation flow on the $d$-torus, is either uniformly hyperbolic or has zero Lyapunov exponents provided that the components of winding vector $\bar{\gamma} = (\gamma^1, \cdots, \gamma^d)$ of the rotation flow are rationally independent and satisfy the following super Liouvillian condition :

$$|\gamma^i - \frac{p_n^i}{q_n}| \leq Ce^{-q_n^{1+\delta}}, \quad 1 \leq i \leq d\,, n \in \mathbb{N}\,,$$

where $C > 0$ and $\delta > 0$ are some constants and $p_n^i\,, q_n$ are some sequences of integers with $q_n \to \infty$.
**Keywords** Cocycles, Lyapunov exponents, irrational rotations, Liouville numbers
**Mathematics Subject Classification (2000)** 37B55, 34A30, 58F15

## 1 Notation, Basic facts and Statement of the Main Result

R. Mane conjectured that a generic volume preserving $C^1$ diffeomorphism of a compact, connected $C^\infty$ surface is either Anosov or its Lyapunov exponents are zero at almost all points. A first step towards resolving this conjecture is to prove its linearized version-namely a generic $SL(2,\mathbb{R})$ valued $C^0$ cocycle is either uniformly hyperbolic or its Lyapunov exponents are zero at almost all points. In the context of cocycles based on flows (rather than discrete dyamical systems), a result of this sort first appeared in the work of R. Fabbri and R. Johnson (cf. [4]) when the base flow is the irrational winding flow on a $d$ torus. This result describes generic behaviour of a $C^r$, $(0 < r < 1)$ cocycle, however it is also generic in terms of the choice of the winding vector of the base rotation flow. In this paper we shall improve this result by fixing the base winding vector. We shall impose a 'super Liouville type' condition on the base winding vector. In fact we prove such a result in a 'much thinner' class of cocycles arising as a fundamental matrix solution to linear differential systems of special form. More precisely, linear differential systems satisfying a ASP ('admissible spectral parameter') property. In particular it can be applied to the class of Schrödinger cocycles.

Some historical remarks are in order. A few years ago, in the setting of discrete dynamical systems J. Bochi proved the above cocycle genericity result for general base transformations in the $C^0$ category, (cf. [1]). This work subsequently led J. Bochi and M. Viana to prove Mane's conjecture (cf. [2]).

As mentioned in their work, this result is not valid in the category of $C^1$ cocycles for general base transformations, for example linear automorphisms of the two torus. We do not know whether such a generic result holds in $C^1$ category when the base flow is a minimal rotation flow.

Even though the results of Bochi-Viana and Fabbri-Johnson have the same flavour, the techniques employed are completely different. Fabbri-Johnson heavily use properties of the 'rotation number' of a cocycle. On the other hand, the Bochi-Viana argument is based on 'switching the stable and unstable directions' of the underlying invariant subbundles. We now begin with basic definitions, facts and the notation.

**Definition 1.1** *A flow $(\Omega, \{T_t\}_{t\in\mathbb{R}})$ consists of a compact metric space $\Omega$ and a one parameter group $\{T_t\}_{t\in\mathbb{R}}$ of homeomorphisms of $\Omega$ such that the action $(\omega, t) \to T_t(\omega) \in \Omega$ is jointly continuous. If $\Omega$ is a $C^\infty$ manifold and the action is jointly $C^k$, $(k \in \mathbb{N})$, then the flow is said to be a $C^k$ flow.*

**Example 1.2** *Our prime example of a flow is the rotation flow on the d-torus-$\mathcal{T}^d$. We shall think of $\Omega = \mathcal{T}^d$ as the unit cube whose opposite faces are identified. If $\bar{\gamma}, \bar{\xi} \in \Omega$, the metric $|\ |$ on $\Omega$ is given by setting $|\bar{\gamma} - \bar{\xi}| = \max_{1\le i\le d} |\gamma^i - \xi^i|$, where $\gamma^i$ and $\xi^i$ are the respective $i^{th}$ components. The flow on $\Omega$ is given by the rule*

$$T_t^{\bar{\gamma}}(x_1, \cdots, x_d) = (x_1 + \gamma^1 t, \cdots, x_d + \gamma^d t)\,, \tag{1.1}$$

*where $\bar{\gamma} = (\gamma^1, \cdots, \gamma^d) \in \mathbb{R}^d$ is the winding vector of the flow.*

First we shall recall basic definitions and facts. Given a rotation flow $(\Omega, \{T_t^{\bar{\gamma}}\}_{t\in\mathbb{R}})$, let $0 \le r < 1$ and consider the space $C^r(\Omega, sl(2,\mathbb{R}))$ of $C^r$ functions from $\Omega$ to $SL(2,\mathbb{R})$. We recall that $A \in C^r(\Omega, sl(2,\mathbb{R}))$ if and only if $||A||_r$-the $C^r$ norm of $A$-is finite. The $C^r$ norm is defined by setting

$$||A||_r = ||A||_0 + \sup_{\omega\neq\eta} \frac{||A(\omega) - A(\eta)||}{|\omega - \eta|^r}\,,$$

where $|\ |$ is the metric on the torus defined before, where the norm on $sl(2,\mathbb{R})$ is the usual operator norm, where the metric on $\mathbb{R}^2$ is the standard Euclidean one, and where $||\ ||_0$ denotes the supremum norm on $C^0(\Omega, sl(2,\mathbb{R}))$.

Given a function $A \in C^r(\Omega, sl(2,\mathbb{R}))$, consider the family of linear differential equations parametrized by points of $\Omega$:

$$x' = A(T_t^{\bar{\gamma}}\omega)x, \qquad x \in \mathbb{R}^2,\ \omega \in \Omega\,. \tag{1.2}$$

For each $\omega \in \Omega$, let $t \to X_A^{\bar{\gamma}}(\omega, t)$ be the fundamental matrix solution of the equation (1.2) satisfying $x(0) = I$-the $2\times 2$ identity matrix. Then the map $X_A^{\bar{\gamma}} : \Omega \times \mathbb{R} \to SL(2,\mathbb{R})$ is a cocycle, i.e. it is continuous and satisfies the following cocycle identity:

$$X_A^{\bar{\gamma}}(\omega, t+s) = X_A^{\bar{\gamma}}(T_t(\omega), s) X_A^{\bar{\gamma}}(\omega, t) \qquad \text{for all } \omega \in \Omega,\ t, s \in \mathbb{R}\,. \tag{1.3}$$

**Lyapunov exponents and Exponential dichotomy**

Next, we recall the definition of the (largest) Lyapunov exponent of a cocycle.

**Definition 1.3** *Let $X_A^{\bar{\gamma}} : \Omega \times \mathbb{R} \to SL(2,\mathbb{R})$ be a cocycle. Let*

$$\beta(X_A^{\bar{\gamma}}) = \lim_{t\to\infty} \frac{1}{t} ln||X_A^{\bar{\gamma}}(\omega, t)||\,.$$

*The Oseledets multiplicative ergodic theorem guarantees the existence of this limit a.e. $\omega$ with respect to the Lebesgue measure $\mu$ on $\Omega$.*

Another way to capture the logarithmic growth rate of the solutions is through the notion of uniform hyperbolicity (or exponential dichotomy).

**Definition 1.4** *A cocycle* $X_A^{\bar{\gamma}} : \Omega \times \mathbb{R} \to SL(2,\mathbb{R})$ *is said to have an exponential dichotomy (ED) if there are constants* $C\,,\rho > 0$ *and a continuous projection valued map* $P : \Omega \to Proj(\mathbb{R}^2) : \omega \to P_\omega$ *such that*

$$||X_A^{\bar{\gamma}}(\omega,t)P_\omega X_A^{\bar{\gamma}}(\omega,s)^{-1}|| \le Ce^{-\rho(t-s)} \quad \text{if } t \ge s\,, \tag{1.4}$$

$$||X_A^{\bar{\gamma}}(\omega,t)(I-P_\omega)X_A^{\bar{\gamma}}(\omega,s)^{-1}|| \le Ce^{\rho(t-s)} \quad \text{if } t \le s\,. \tag{1.5}$$

*The set* $\Sigma(A)$ *defined by*

$$\Sigma(A) = \{\lambda \in \mathbb{R} \mid \text{the cocycle } e^{-\lambda t}X_A^{\bar{\gamma}}(\omega,t) \text{ does not admit an ED}\},$$

*is called the dichotomy spectrum of the cocycle* $X_A^{\bar{\gamma}}$.

For the rotation flow whose winding vector has rationally independent components, there are exactly three possibilities described in the following proposition (cf. [8]).

**Proposition 1.5** *For a minimal rotation flow (or more generally for any minimal uniquely ergodic flow), given a cocycle* $X_A$ *into* $SL(2,\mathbb{R})$*, the dichotomy spectrum* $\Sigma(A)$ *is either a singleton set* $\{0\}$ *or a two point set* $\{-\beta(X_A)\,,\beta(X_A)\}$ *or the interval* $[-\beta(X_A)\,,\beta(X_A)]$*. The first case is equivalent to saying that the cocycle* $X_A$ *admits an ED, and the second case holds exactly when* $\beta(X_A) = 0$*-which we shall refer to as the zero exponent case.*

The dichotomy property is closely related to the spectral properties of certain differential operators. This relation results with an introduction of a 'spectral parameter' $\lambda$. This is done as follows: given $A \in C^0(\Omega, sl(2,\mathbb{R}))$, consider the system

$$x' = [A(T_t^{\bar{\gamma}}\omega) + \lambda J]x\,, \tag{1.6}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\lambda \in \mathbb{R}$ or more generally in $\mathbb{C}$. The system generates the cocycle $X_{A+\lambda J}^{\bar{\gamma}}$. Also consider the following AKNS operator $L_\omega$ associated with $A$,

$$L_\omega = J^{-1}\big[\frac{d}{dt} - A(T_t^{\bar{\gamma}}\omega)\big]\,.$$

This is viewed as an unbounded self-adjoint operator on the Hilbert space $L^2(\mathbb{R},\mathbb{C}^2)$ of square integrable $\mathbb{C}^2$-valued functions on $\mathbb{R}$. The following proposition describes the resolvent of this operator in terms of the notion of ED, (cf. [6]).

**Proposition 1.6** *A complex number* $\lambda$ *is in the resolvent of* $L_\omega$ *if and only if the cocycle* $X_{A+\lambda J}^{\bar{\gamma}}$ *admits ED.*

**Property ASP**

We want to study the generic behaviour of cocycles arising from differential systems of a special form (for example those arising from the Schrödinger equation). The 'special form' of the equation is a constraint which we shall capture by restricting the function $A : \Omega \to sl(2,\mathbb{R})$ to take values in a preassigned subset $S \subseteq sl(2,\mathbb{R})$. It turns out that if $S$ has certain properties then the crucial 'rotation number argument', (described in the next section) can be carried out in the class $C^r(\Omega, S)$ of $S$ valued functions instead of the much bigger class $C^r(\Omega, sl(2,\mathbb{R}))$. In the following definition we abstract the features of $S$ we need to do this.

**Definition 1.7** *A subset $S \subseteq sl(2,\mathbb{R})$ has the property ASP ('admissible spectral parameter' property) if it is compact, convex, non-singleton and $V_S \cap \{A \in sl(2,\mathbb{R}) \mid det(A) > 0\} \neq \emptyset$, where $V_S$ is the linear subspace of $sl(2,\mathbb{R})$ spanned by $S$.*

It is easy to verify that, in the following examples, the set $S$ has ASP.

**Example 1.8** *(Schrödinger system) Fix $\lambda \in \mathbb{R}$ and let $S \equiv S_\lambda = \{\begin{pmatrix} 0 & 1 \\ r-\lambda & 0 \end{pmatrix} \mid r \in F\}$, where $F \subset \mathbb{R}$ is a closed convex, non-singleton subset of $\mathbb{R}$. Then it is easy to verify that $V_S = \{\begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix} \mid \mu \in \mathbb{R}\}$ and $S$ has ASP.*

**Example 1.9** *(Bylov - Vinograd system) Let $F \subset \mathbb{R}$ is a closed convex, non-singleton subset of $\mathbb{R}$. Set, $S = \{\begin{pmatrix} 0 & 1-r \\ 1+r & 0 \end{pmatrix} \mid r \in F\}$. Again, it is easy to verify that $S$ has ASP.*

**Statement of the Theorem**

Now we shall state the main theorem precisely.

**Theorem 1.10** *Consider the rotation flow $(\Omega, \{T_t^{\bar{\gamma}}\}_{t\in\mathbb{R}})$ on the $d$-torus $\Omega$, where the winding vector $\bar{\gamma} = (\gamma^1, \cdots, \gamma^d)$ satisfies the following (super Liouville) condition : the components of $\bar{\gamma}$ are rationally independent and satisfy the following super Liouvillian condition :*

$$|\bar{\gamma} - \bar{\gamma}_n| \leq Ce^{-q_n^{1+\delta}}, \quad n \in \mathbb{N}, \tag{1.7}$$

*where $C > 0$ and $\delta > 0$ are some constants and $\bar{\gamma}_n = (\frac{p_n^1}{q_n}, \cdots, \frac{p_n^d}{q_n})$ for sequences $\{p_n^i\}$, $(1 \leq i \leq d)$, $\{q_n\}$ in $\mathbb{N}$ and $q_n \to \infty$. Let $S \subseteq sl(2,\mathbb{R})$ be a subset with property ASP. Let $0 < r < 1$. Consider the set*

$$\mathcal{C}_M(S) = \{A \in C^r(\Omega, S) \mid \textit{either } X_A^{\bar{\gamma}} \textit{ is uniformly hyperbolic or } \lambda(X_A^{\bar{\gamma}}) = 0\}.$$

*Then $\mathcal{C}_M$ is residual in $C^r(\Omega, S)$.*

## 2 The Rotation Number Argument

In this section we give a brief introduction of the rotation number of a cocycle, state its basic properties and describe the 'rotation number argument' which proves Proposition (2.4). This proposition plays a crucial role in our proof. The flow $(\Omega, \{T_t^{\bar{\gamma}}\}_{t\in\mathbb{R}})$ is the rotation flow on the $d$ torus with rotation vector $\gamma$. Consider the family

$$x' = A(T_t^{\bar{\gamma}}(\omega))x, \quad \omega \in \Omega, \quad t \in \mathbb{R},$$

where $A = \begin{pmatrix} d & -b+c \\ b+c & -d \end{pmatrix}$ and $b, c, d$ are continuous real valued functions of $\omega$. Introducing the usual polar co-ordinates $(r, \theta)$, the above linear equation can be written as

$$r' = \left[d(T_t^{\bar{\gamma}}\omega)cos(2\theta) + c(T_t^{\bar{\gamma}}(\omega)sin(2\theta)\right]r \tag{2.1}$$

$$\theta' = b(T_t^{\bar{\gamma}}\omega) + c(T_t^{\bar{\gamma}}\omega)cos(2\theta) - d(T_t^{\bar{\gamma}}\omega)sin(2\theta)\,. \tag{2.2}$$

Note that the $\theta$ equation does not depend on $r$.

**Definition 2.1** *The rotation number $\alpha \equiv \alpha(\omega, \bar{\gamma}, A)$ of the above family is defined by setting*

$$\alpha = \lim_{t\to\infty} \frac{\theta(t)}{t}\,,$$

*where $\theta(t)$ is a solution of the $\theta$ equation with arbitrary initial condition $\theta(0) = \theta_0$.*

We list the basic properties of $\alpha$, (cf. [6] and [7]) for details).

**Rotation Number : Continuity Properties**

(**1**) The above limit exists and is independent of the initial condition $\theta_0$. Furthermore, if the flow is minimal it is independent of $\omega$.
(**2**) For each fixed $\bar{\gamma} \in \mathbb{R}^d$ the map $(\omega, A) \to \alpha(\omega, \bar{\gamma}, A) : \Omega \times C^0(\Omega, sl(2, \mathbb{R})) \to \mathbb{R}$ is continuous.
(**3**) If $\bar{\gamma}$ has rationally independant components and if $\bar{\gamma}_n \in \mathbb{R}^d$ is such that $\bar{\gamma}_n \to \bar{\gamma}$, then $\alpha(\omega, \bar{\gamma}_n, A) \to \alpha(\omega, \bar{\gamma}, A)$ where the convergence is uniform on compact subsets of $\Omega \times C^0(\Omega, sl(2, \mathbb{R}))$.

**Rotation Number : Spectral Properties**

(**4**) The following theorem relates rotation number of a cocycle to the spectrum of the associated AKNS operator.

**Theorem 2.2** *Let $A_0 \in C^0(\Omega, sl(2, \mathbb{R}))$ and suppose $\bar{\gamma}$ has rationally independent components. Suppose that the cocycle $X^{\bar{\gamma}}_{A+\lambda J}$ generated by the system (1.6) does not admit ED for all $\lambda \in I$, where $I \subset \mathbb{R}$ is some open interval containing 0. Then the map $\lambda \to \alpha(\omega, \bar{\gamma}, A_0 + \lambda J)$ is strictly increasing.*

(**5**) We shall also need the following fact about periodic AKNS operators.

**Proposition 2.3** *Suppose the flow $(\Omega, \{T_t^{\bar{\gamma}}\}_{t\in\mathbb{R}})$ is periodic with period $T > 0$. Let $A \in C^0(\Omega, sl(2, \mathbb{R}))$.*
*(I) Suppose $I \subset \mathbb{R}$ is an open interval such that if $\lambda \in I$ then $X^{\bar{\gamma}}_{A+\lambda J}$ does not admit ED on the orbit closure of some $\omega \in \Omega$. Then the map $\lambda \to \alpha(\omega, \bar{\gamma}, A + \lambda J)$ is analytic.*
*(II) Suppose for some $\omega \in \Omega$ and $\lambda \in \mathbb{R}$, $\alpha(\omega, \bar{\gamma}, A + \lambda J) = \frac{2\pi m}{T}$ for some integer $m$. Then $\lambda$ is either (i) in the closure of a resolvent interval or (ii) $\lambda$ is a 'closed gap'. In this last case, $tr(X^{\bar{\gamma}}_{A+\lambda J}(\omega, T)) = 2$ because the rotation number is an even multiple of $\frac{\pi}{T}$.*

We say that $\lambda \in \mathbb{R}$ is a closed gap for the operator $L_\omega = J^{-1}\left[\frac{d}{dt} - A(T_t^{\bar{\gamma}}\omega)\right]$ if the rotation number $\alpha(\omega, \bar{\gamma}, A + \lambda J) = \frac{\pi k}{T}$ for some integer $k$ and if $\lambda$ is in the interior of the spectrum of $L_\omega$. Now we shall state the crucial result we need.

**Proposition 2.4** *Let $A_0 \in C^0(\Omega, sl(2,\mathbb{R}))$. Suppose $\bar{\gamma}$ has rationally independent components and fix a sequence $\bar{\gamma}_n \to \bar{\gamma}$ so that the flow $(\Omega, \{T_t^{\bar{\gamma}_n}\}_{t\in\mathbb{R}})$ is periodic with period $q_n$ and $q_n \to \infty$. Suppose that for some $\eta > 0$, the cocycle $X_A^{\bar{\gamma}}$ does not admit ED for any $A$ such that $||A - A_0||_r < \eta$.*
*(I) Then given any $\varepsilon > 0$, there exists some $n_1 \in \mathbb{N}$ such that for each $n \geq n_1$ there exists a function $A_n \in C^r(\Omega, sl(2,\mathbb{R}))$ such that*

*(1) $||A - A_n||_r < \varepsilon$ and*

*(2) $tr\big(X_{A_n}^{\bar{\gamma}_n}(\omega, q_n)\big) = 2$, for all $\omega \in \Omega$.*

*(II) Furthermore, if $A_0 \in C^r(\Omega, S)$, where $S \subset sl(2,\mathbb{R})$ has ASP, then the functions $A_n$ can be selected to be in $C^r(\Omega, S)$.*

**Proof:** (I) We only sketch the argument, for complete details see (cf. [4]) and (cf. [5]). From the hypothesis it follows that there is some finite, open interval $I \subset \mathbb{R}$ containing zero such that if $\lambda \in I$ then the cocycle $X_{A_0+\lambda J}^{\bar{\gamma}}$ does not admit ED, (and without loss of generality, we shall let $|I| < \varepsilon$). Hence the function $\lambda \to \alpha(\omega, \bar{\gamma}, A_0+\lambda J)$ is strictly increasing on $I$. Since (i) as $n \to \infty$, $\alpha(\omega, \bar{\gamma}_n, A_0) \to \alpha(\omega, \bar{\gamma}, A_0)$, uniformly on $\Omega \times \bar{I}$, and (ii) for each $n \in \mathbb{N}$ the map $\lambda \to \alpha(\omega, \bar{\gamma}_n, A_0 + \lambda J)$ is continuous (uniformly in $\omega \in \Omega$), we can find some $n_1 \in \mathbb{N}$ with the following property : for each $n \geq n_1$, and for any $\omega \in \Omega$ there is some integer $\ell$, (which depends on $n$ and $\omega$) such that

$$(\frac{2\pi(\ell-1)}{q_n}, \frac{2\pi(\ell+1)}{q_n}) \subseteq \{\alpha(\omega, \bar{\gamma}_n, A_0 + \lambda J) \mid \lambda \in I\}\,.$$

Fix any $n \geq n_1$. Then for each $\omega \in \Omega$ let $\lambda_n(\omega)$ be the smallest value of $\lambda$ such that $\alpha(\omega, \bar{\gamma}_n, A_0+\lambda J) = \frac{2\pi\ell}{q_n}$. Using monotonicity of $\alpha$ in $\lambda$ one can check that the map $\omega \to \lambda_n(\omega) : \Omega \to I$ is well defined and continuous. Since $|I| < \varepsilon$, letting $A_n = A_0+\lambda_n(\omega)J$, we get $||A_n - A||_0 < \varepsilon$. Since $\{T_t^{\bar{\gamma}_n}\}_{t\in\mathbb{R}}$ is a periodic flow with period $q_n$ and $\alpha(\omega, \bar{\gamma}_n, A_n) = \frac{2\pi\ell}{q_n}$, it follows (from Proposition (2.3)) that $tr\big(X_{A_n}^{\bar{\gamma}_n}(\omega, q_n)\big) = 2$, for all $\omega \in \Omega$. This proves the first part of above proposion in the '$C^0$ category'. To prove the same in $C^r$ category requires more effort. In the following we sketch this argument.

To establish regularity (i.e. 'smoothness properties') of the map $\omega \to \lambda_n(\omega)$ obtained above, we need to construct it more carefully. First, if necessary, replacing $A_0$ by a close enough $C^1$ function, we shall assume that $A_0$ is $C^1$. As before, fix any $n \geq n_1$. For each $\omega \in \Omega$, define the interval $I_{n,\omega} \subset I$ by setting

$$I_{n,\omega} = \{\lambda \in I \mid \alpha(\omega, \bar{\gamma}_n, A_0 + \lambda J) \in (\frac{2\pi(\ell - \frac{1}{2})}{q_n}, \frac{2\pi\ell}{q_n})\}\,.$$

Note that $I_{n,\omega}$ is non-degenerate and the map $\lambda \to \alpha(\omega, \bar{\gamma}_n, A_0 + \lambda J)$ is analytic on it and is $C^1$ in $\omega$. Hence by the implicit function theorem for each $b \in (\frac{2\pi(\ell-\frac{1}{2})}{q_n}, \frac{2\pi\ell}{q_n})$ the equation $\alpha(\omega, \bar{\gamma}_n, A_0 + \lambda J) = b$ has a unique solution $\lambda_b(\omega)$, which is $C^1$ in $\omega$. Note that $\lambda_b(\omega) \to \lambda_n(\omega)$ as $b$ increases to $\frac{2\pi\ell}{q_n}$, where $\lambda_n(\cdot)$ is the map obtained in the first paragraph. Let $A_b = A_0 + \lambda_b(\omega)J$. Our desired function $A_n$ is then the limit (-pointwise in $\omega$-) of $A_b$ as $b$ increases to $\frac{2\pi\ell}{q_n}$. It is in the process of taking this limit, that $C^1$ smoothness of $\lambda_b$ is lost.

Now the key step is to get a uniform bound on the Lipschitz constants of the maps $\lambda_b$, where $b \in (\frac{2\pi(\ell-\frac{1}{2})}{q_n}, \frac{2\pi\ell}{q_n})$. We shall show that $\mathrm{Lip}(\lambda_b) \leq ||A_0||_1$. Assuming this, we complete the proof as follows. This estimate clearly implies that $\mathrm{Lip}(\lambda_n) \leq ||A_0||_1$. Now for each $0 \leq r < 1$ one has

$$||\lambda_n||_r \leq ||\lambda_n||_0(1 + (\mathrm{Lip}(\lambda_n))^r 2^{1-r})\,.$$

Since the Lipschitz constants of $\lambda_n$'s are uniformly bounded above by $||A_0||_1$, above estimate shows that $||\lambda_n||_r$ can be made as small as desired by making sure that $||\lambda_n||_0$ is small enough. If we pick $|I| < \frac{\varepsilon}{1+||A_0||_1^r 2^{1-r}}$, then $A_n = A_0 + \lambda_n$ is a $C^r$ function with $||A_n - A_0||_r < \varepsilon$ and the proof of the proposition is complete.

**Regularity of $\lambda_b$-an argument based on J. Moser's computation**

Now we sketch the proof of the estimate $\mathrm{Lip}(\lambda_b) \le ||A_0||_1$. Recall that for each $b \in (\frac{2\pi(\ell-\frac{1}{2})}{q_n}, \frac{2\pi\ell}{q_n})$, the functions $\lambda_b : \Omega \to \mathbb{R}$ were obtained as solutions to the equation

$$\alpha(\omega, \bar{\gamma}_n, A_0 + \lambda_b(\omega)J) = b\,.$$

Differentiating this relation with respect to components of $\omega = (\omega_1, \cdots, \omega_d)$ yields

$$0 = \frac{\partial\alpha}{\partial\omega_j} + \frac{\partial\alpha}{\partial\lambda}\frac{\partial\lambda_b}{\partial\omega_j}\,.$$

Thus to control $\frac{\partial\lambda_b}{\partial\omega_j}$, we need to control $\frac{\partial\alpha}{\partial\omega_j}$ and $\frac{\partial\alpha}{\partial\lambda}$. First we describe how, for periodic AKNS operators these partial derivatives are related to partial derivatives of its 'discriminant function'.

We describe this relation in a more general context. Let $T > 0$ be given and let $C_T$ denote the space of continuous $T$-periodic $sl(2,\mathbb{R})$ valued functions on $\mathbb{R}$. Given $a \in C_T$, let $X_a : \mathbb{R} \to SL(2,\mathbb{R})$ denote the fundamental matrix solution to the equation $x' = a(t)x$ with initial condition $x(0) = I$-the identity matrix. Define the discriminant functional $\hat{\Delta} : C_T \to \mathbb{R}$ by setting

$$\hat{\Delta}(a) = tr(X_a(T))\,.$$

J. Moser's computation refers to the computation of the Frechet derivative of $\hat{\Delta}$ at a point $a \in C_T$ with the property that $X_a(T)$ is conjugate to a rotation matrix. The relation of this to our regularity problem goes as follows.

Consider the periodic flow $(\Omega, \{T_t^{\bar{\gamma}_n}\}_{t\in\mathbb{R}})$ with period $T = q_n$, (recall that $n \ge n_1$ is fixed). Given $\omega \in \Omega$ and $A \in C^1(\Omega, sl(2,\mathbb{R}))$, set $a_{\omega,A} = A(T_t^{\bar{\gamma}_n}\omega)$, then $a_{\omega,A} \in C_T$. Now fix $A_0 \in C^1(\Omega, sl(2,\mathbb{R}))$ and define $\Delta : \Omega \times \mathbb{R} \to \mathbb{R}$ by setting

$$\Delta(\omega, \lambda) = \hat{\Delta}(a_{\omega, A_0+\lambda J})\,.$$

Let $\Sigma_{A_0}$ be the spectrum of the AKNS operator $L_\omega = J^{-1}\left[\frac{d}{dt} - A_0(T_t\omega)\right]$ on $C_{q_n}$. Since $b \in (\frac{2\pi(\ell-\frac{1}{2})}{q_n}, \frac{2\pi\ell}{q_n})$, $\lambda_b(\omega)$ is an interior point of $\Sigma_{A_0}$. Hence, (using the spectral theory of periodic AKNS operators), $X_{a_{\omega,A_0+\lambda J}}(q_n)$ is conjugate to the rotation matrix $\begin{pmatrix} cos(q_n\alpha) & -sin(q_n\alpha) \\ sin(q_n\alpha) & cos(q_n\alpha) \end{pmatrix}$, where $\alpha \equiv \alpha(\omega, \bar{\gamma}_n, A_0+\lambda J)$. This yields the following relation between $\alpha$ and $\Delta$,

$$\Delta(\omega, \lambda) = 2cos(q_n\alpha(\omega, \bar{\gamma}_n, A_0 + \lambda J))\,.$$

Thus

$$\frac{\partial\alpha}{\partial\lambda} = \frac{-1}{2q_n sin\,(q_n\alpha)}\frac{\partial\Delta}{\partial\lambda}\,, \quad \text{and} \quad \frac{\partial\alpha}{\partial\omega_j} = \frac{-1}{2q_n sin\,(q_n\alpha)}\frac{\partial\Delta}{\partial\omega_j}\,.$$

Thus, we can compute $\frac{\partial\lambda_b}{\partial\omega_j}$ in terms of partial derivatives of $\Delta(\omega,\lambda)$. Now note that these partial derivatives of $\Delta$ can be computed if we write down the Frechet derivative of $\hat{\Delta}$ at the point $a_{\omega,A_0+\lambda J}$. This Moser - type computation (cf. [9]) is the content of the following lemma.

**Lemma 2.5** *Consider a basis of $sl(2,\mathbb{R})$ given by matrices*

$$J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

*So any function $a \in C_T$ will be written as $a(t) = \sum_{1=1}^{3} a_i(t) J_i$. Let $a^* \in C_T$ so that $X_{a^*}(T)$ is conjugate to the rotation matrix corresponding to angle $T\alpha$. Then $D_{a^*}(\hat{\Delta})(a)$-the directional derivative (in the direction $a$) of $\hat{\Delta} : C_T \to \mathbb{R}$ at $a$-is given by*

$$D_{a^*}(\hat{\Delta})(a) = -sin(\alpha T) \int_0^T \big(\sum_{i=1}^{3} K_i(s) a_i(s)\big) ds\,,$$

*where $K_i : \mathbb{R} \to \mathbb{R}$ are functions such that for any $t \in [0,T]$, (i) $0 < K_1(t)$ and (ii) $|K_i(t)| \leq K_1(t)$, $i = 2,3$.*

A sketch of the proof : We follow [4], (pp. 366-367) but take the opportunity to correct a minor error in that presentation. First one shows that

$$D_{a^*}(\hat{\Delta})(a) = tr\Big(X_{a^*}(T) \int_0^T X_{a^*}(s)^{-1} a(s) X_{a^*}(s) ds\Big).$$

Next, one writes $X_{a^*}(T) = Q^{-1} R_T Q$, where

$$R_T = \begin{pmatrix} cos(T\alpha) & -sin(T\alpha) \\ sin(T\alpha) & cos(T\alpha) \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

are matrices in $SL(2,\mathbb{R})$. One also writes $X_{a^*}(t) = \begin{pmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{pmatrix}$. For each $i = 1,2,3$ one has

$$\begin{aligned} tr\big(X_{a^*}(T) X_{a^*}(t)^{-1} J_i X_{a^*}(t)\big) &= tr\big(Q^{-1} R_T Q X_{a^*}(t)^{-1} J_i X_{a^*}(t) Q^{-1} Q\big) \\ &= tr\big(R_T Q X_{a^*}(t)^{-1} J_i X_{a^*}(t) Q^{-1}\big) \\ &= tr\big(R_T B J_i B^{-1}\big)\,, \end{aligned}$$

where $B = Q X_{a^*}(t)^{-1}$. Now set $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and check that $tr\big(R_T B J_i B^{-1}\big) = -sin(\alpha T) K_i(t)$, where

$$\begin{aligned} K_1(t) &= b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2\,, \\ K_2(t) &= 2b_{11}b_{12} + 2b_{21}b_{22}\,, \\ K_3(t) &= -b_{11}^2 - b_{21}^2\,. \end{aligned}$$

It follows that $K_1(t) > 0$, $|K_2(t)| \leq K_1(t)$ and $|K_3(t)| \leq K_1(t)$ for all $t \in [0,T]$. One has the following explicit formula :

$$K_1(t) = (u_1^2 + v_1^2)(q_{12}^2 + q_{22}^2) + (u_2^2 + v_2^2)(q_{11}^2 + q_{21}^2) - 2(q_{11}q_{12} + q_{21}q_{22})(u_1u_2 + v_1v_2)\,.$$

With this lemma and the above discussion one can show (cf. [4]) that

$$\frac{\partial \lambda_b}{\partial \omega_j} = \frac{-\sum_{i=1}^{3} \int_0^T K_i(t) \frac{\partial A_0^i}{\partial \omega_j}(T_t^{\bar{\gamma}}(\omega)) dt}{\int_0^T K_1(t) dt}\,.$$

Now the properties of $K_i$'s lead to the estimate $|\frac{\partial\lambda_b}{\partial\omega_j}| \leq ||A_0||_1$.

(II) We recall the hypotheses : $A_0 \in C^r(\Omega, S)$, $S$ has ASP and $X_A^{\bar{\gamma}}$ does not admit ED for any $A \in C^r(\Omega, S)$ with $||A - A_0||_r < \eta$. Since $S$ is non-singleton and convex, $S^0$-the relative interior of $S$ as a subset of $V_S$-is non-empty, (recall that $V_S$ is the subspace of $sl(2,\mathbb{R})$ generated by $S$). Thus, if necessary, replacing $A_0$ by another $S$ valued function that is $C^r$ close to it, without loss of generality we can assume that $A_0$ is $C^1$ and takes values in $S^0$. Since the image $A_0(\Omega)$ is a compact subset of $S^0 \subset V_S$, there exists a (Euclidean) ball $B$ centered at the zero vector of the vector space $V_S$ such that $A_0(\omega) + v \in S^0 \subset S$, for all $v \in B$ and $\omega \in \Omega$. Since $S$ has ASP, $B$ contains a matrix in $sl(2,\mathbb{R})$ with determinant-say $\mu$, ($\mu > 0$). Such a matrix must be of the form $\mu P^{-1}JP$, for some non-singular real matrix $P$, (where $J$ is as before). Thus there is some open interval $I$ containing 0 such that $A_0 + \lambda\mu P^{-1}JP \in S$ for all $\lambda \in I$. In other words there is a $\rho > 0$ such that $A_0 + \lambda P^{-1}JP \in S$, for all $|\lambda| < \rho$.

Now if $||\tilde{A} - PA_0P^{-1}||_r < \frac{\eta}{||P|| \, ||P^{-1}||}$, then $||P^{-1}\tilde{A}P - A_0||_r < \eta$. Hence by the hypothesis $P^{-1}\tilde{A}P$ does not admit ED. Thus, $\tilde{A}$ does not admit ED. Now apply part (I) of this Proposition to the function $PA_0P^{-1}$ with $\varepsilon$ replaced by $min\{\rho\,, \frac{\varepsilon}{||P|| \, ||P^{-1}||}\}$. Thus, there exists $n_1 \in \mathbb{N}$ and a sequence of functions $\tilde{A}_n$, $(n \geq n_1)$ such that (i) $||\tilde{A}_n - PA_0P^{-1}|| < \frac{\varepsilon}{||P|| \, ||P^{-1}||}$ and (ii) $tr(X_{\tilde{A}_n}^{\bar{\gamma}_n}(\omega, q_n)) = 2$ for all $\omega \in \Omega$. It follows from the proof of part (I) that each $\tilde{A}_n$ is of the form $\tilde{A}_n = PA_0P^{-1} + \lambda_n(\omega)J$. Take $A_n = P^{-1}\tilde{A}_nP$, then

$$||A_n - A_0|| \leq ||P|| \, ||P^{-1}\big(\tilde{A}_n - PA_0P^{-1}\big)P|| \; ||P^{-1}|| < \varepsilon \quad \text{and}$$

$$tr(X_{A_n}^{\bar{\gamma}}(\omega, q_n)) = tr(P^{-1}X_{\tilde{A}_n}^{\bar{\gamma}}(\omega, q_n)P) = tr(X_{\tilde{A}_n}^{\bar{\gamma}}(\omega, q_n)) = 2\,.$$

Finally we need to show that each $A_n$ is $S$ valued. Note that $A_n(\omega) = P^{-1}\tilde{A}_nP = A_0 + \lambda_n(\omega)P^{-1}JP \in S$ for all $\omega \in \Omega$, because we have made sure that $|\lambda_n(\omega)| < \rho$. This concludes the proof. ∎

# 3 The proof of Theorem (1.10)

**Proof:** For each $k \in \mathbb{N}$ consider the set

$$V(k) = \{A \in C^r(\Omega, S) \mid X_A^{\bar{\gamma}} \text{ is not uniformly hyperbolic and } \lambda(X_A^{\bar{\gamma}}) \geq \frac{1}{k}\}\,.$$

Since $S$ is closed, the upper semi-continuity of the Lyapunov exponent implies that $V(k)$ is a closed set. We will show that each $V(k)$ has empty interior. This will imply that $\cup_{k\in\mathbb{N}}V(k)$ is a set of first category. Then the proof follows from the Baire Category theorem since $\mathcal{C}_M \subset C^r(\Omega, S) \setminus \cup_{k\in\mathbb{N}}V(k)$.

So let us suppose that for some $k \in \mathbb{N}$, $V(k)$ has non empty interior. This $k$ will be fixed here onwards. Observe that

$$\lambda(X_A^{\bar{\gamma}}) = \lim_{t\to\infty} \frac{1}{t} \int_\Omega ln\,||X_A^{\bar{\gamma}}(\omega, t)||d\omega \;= \inf\nolimits_{t>0}\{\frac{1}{t}\int_\Omega ln\,||X_A^{\bar{\gamma}}(\omega, t)||d\omega\}\,,$$

where we have used the subadditivity of the map $t \to \int_\Omega ln\,||X_A^{\bar{\gamma}}(\omega, t)||d\omega$. Thus if $A \in V(k)$, then $\int_\Omega ln\,||X_A^{\bar{\gamma}}(\omega, t)||d\omega \geq \frac{1}{k}$ for all $t \geq 0$. Our assumption implies that there is some $A_0 \in V(k)$ and some $\eta > 0$ such that $B_\eta(A_0) \equiv \{B \in C^r(\Omega, S) \mid ||B - A_0||_r < \eta\} \subseteq V(k)$. Without loss of generality, we shall let $\eta < ||A_0||_r$. Thus,

$$||B||_0 \leq ||B||_r \leq 2||A_0||_r\,, \quad \text{if } B \in B_\eta(A_0)\,.$$

In the following, we shall construct a function $B \in B_\eta(A_0)$ such that $B \notin V(k)$. This contradiction will complete the proof.

Step (1) : (The Rotation Number argument) : Since $X_A^{\bar{\gamma}}$ does not admit ED for any $A \in B_\eta(A_0)$, we can apply Proposition (2.4) with $\varepsilon = \frac{\eta}{2}$ and with $\bar{\gamma}_n = (\frac{p_n^1}{q_n}, \cdots, \frac{p_n^d}{q_n})$ (-the sequence in the definition of the super Liouville condition). In this way we get $n_1 \in \mathbb{N}$ and a sequence of functions $A_n \in C^r(\Omega, S)$ such that

(1) $||A - A_n||_r < \frac{\eta}{2}$ and

(2) $tr\big(X_{A_n}^{\bar{\gamma}_n}(\omega, q_n)\big) = 2$, for all $\omega \in \Omega$.

Step (2) : (Gronwall's Inequality) Let $B \in B_\eta(A_0)$. Our choice of $\eta$ implies that $||B||_0 \leq 2||A_0||_r$. Let $\bar{\xi} \in [0,1]^d$ be any winding vector. Note that $x(t) = X_B^{\bar{\xi}}(\omega, t)x_0$ satisfies the integral equation $x(t) = x_0 + \int_0^t B(T_s^{\bar{\xi}}\omega)x(s)ds$. Hence the estimate $||x(t)|| \leq ||x_0|| + \int_0^t ||B(T_s^{\bar{\xi}}\omega)||\,||x(s)||ds$ allows one to apply Gronwall's inequality which yields the estimate $||X_B^{\bar{\xi}}(\omega, t)x_0|| \leq ||x_0||e^{||B||_0 t} \leq ||x_0||e^{2||A_0||_r t}$ for any $t > 0$. Taking the supremum over all $||x_0||$'s of norm one yields

$$||X_B^{\bar{\xi}}(\omega, t)|| \leq e^{2||A_0||_r t}\,. \tag{3.1}$$

Next we compare the solutions $x^{\bar{\gamma}}(t) \equiv X_B^{\bar{\gamma}}(\omega, t)x_0$ and $x^{\bar{\gamma}_n}(t) \equiv X_B^{\bar{\gamma}_n}(\omega, t)x_0$, where $B \in B_\eta(A_0)$ and $x_0$ is any vector of unit norm. Again, using the integral representation we have

$$x^{\bar{\gamma}}(t) - x^{\bar{\gamma}_n}(t) = \int_0^t B(T_s^{\bar{\gamma}}\omega)x^{\bar{\gamma}}(s)ds - \int_0^t B(T_s^{\bar{\gamma}_n}\omega)x^{\bar{\gamma}_n}(s)ds\,.$$

Thus,

$$||x^{\bar{\gamma}}(t) - x^{\bar{\gamma}_n}(t)|| \leq \int_0^t ||B(T_s^{\bar{\gamma}}\omega) - B(T_s^{\bar{\gamma}_n}\omega)||\,||x^{\bar{\gamma}}(s)||ds + \int_0^t ||B(T_s^{\bar{\gamma}_n}\omega)||\,||x^{\bar{\gamma}}(s) - x^{\bar{\gamma}_n}(s)||ds\,.$$

Now note that for any $s \in [0, T]$ we have

$$\begin{aligned}\sup_{\omega\in\Omega}||B(T_s^{\bar{\gamma}}\omega) - B(T_s^{\bar{\gamma}_n}\omega)|| &\leq ||B||_r \sup_{\omega\in\Omega}|T_s^{\bar{\gamma}}(\omega) - T_s^{\bar{\gamma}_n}(\omega)|^r \\ &\leq 2||A_0||_r \left|\bar{\gamma} - \bar{\gamma}_n\right|^r s^r\,, \quad (\text{since } ||B||_r \leq 2||A_0||_r)\,.\end{aligned}$$

Next, note that the estimate (3.1) gives

$$||x^{\bar{\gamma}}(s)|| = ||X_B^{\bar{\gamma}}(\omega, s)x_0|| \leq e^{2||A||_r s}||x_0|| = e^{2||A||_r s}\,.$$

Thus, for any $t \in [0, T]$ we have

$$||x^{\bar{\gamma}}(t) - x^{\bar{\gamma}_n}(t)|| \leq 2||A_0||_r \left|\bar{\gamma} - \bar{\gamma}_n\right|^r \int_0^t s^r e^{2||A_0||_r s}ds + \int_0^t ||B||_r\,||x^{\bar{\gamma}}(s) - x^{\bar{\gamma}_n}(s)||ds \tag{3.2}$$

We note that $s^r \leq 1$ if $s \in [0,1]$ and $s^r < s$ if $s > 1$. Hence if $t \geq 1$, we have

$$\begin{aligned}\int_0^t s^r e^{2||A_0||_r s}ds &\leq \int_0^1 e^{2||A_0||_r s}ds + \int_1^t s e^{2||A_0||_r s}ds \\ &\leq \frac{t}{2||A_0||_r}e^{2||A_0||_r t} - \int_1^t \frac{e^{2||A_0||_r s}}{2||A_0||_r}ds \\ &\leq \frac{t}{2||A_0||_r}e^{2||A_0||_r t}\,.\end{aligned}$$

Thus, for $t \geq 1$ we have the estimate

$$||x^{\bar{\gamma}}(t) - x^{\bar{\gamma}_n}(t)|| \leq 2||A_0||_r \left|\bar{\gamma} - \bar{\gamma}_n\right|^r \frac{t}{2||A_0||_r} e^{2||A_0||_r t} + 2\int_0^t ||A_0||_r \, ||x^{\bar{\gamma}}(s) - x^{\bar{\gamma}_n}(s)|| ds \,. \tag{3.3}$$

Thus Gronwall's inequality implies that if $t \in [1, T]$ then

$$||x^{\bar{\gamma}}(t) - x^{\bar{\gamma}_n}(t)|| \leq \left|\bar{\gamma} - \bar{\gamma}_n\right|^r T e^{2||A_0||_r T} e^{2||A_0||_r t} \,.$$

Taking the supremum over all $x_0$'s with unit norm yields: if $t \in [1, T]$ and $\omega \in \Omega$ then

$$||X_B^{\bar{\gamma}}(\omega, t) - X_B^{\bar{\gamma}_n}(\omega, t)|| \leq \left|\bar{\gamma} - \bar{\gamma}_n\right|^r T e^{4||A_0||_r T} \,. \tag{3.4}$$

Step (3) : Going back to Step (1), recall that $tr(X_{A_n}^{\bar{\gamma}_n}(\omega, q_n)) = 2$ for all $\omega \in \Omega$ and for each $n \geq n_1$. Now we recall a general fact : if $g \in SL(2, \mathbb{R})$ is a matrix such that $tr(g) = 2$ then $g$ can be written as

$$g = R(\phi) \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} R(\phi)^{-1} \,,$$

for some $\phi \in [0, \pi)$ and $\mu \in \mathbb{R}$, where $R(\phi) = \begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix}$. To see this, note that the characteristic equation of $g$ is $\lambda^2 - 2\lambda + 1 = 0$ and $g$ has a unique eigenvalue 1 with algebraic multiplicity 2. Let $v = \begin{pmatrix} cos(\phi) \\ sin(\phi) \end{pmatrix}$ be an eigenvector of $g$ of standard Euclidean norm 1, and let $w = \begin{pmatrix} -sin(\phi) \\ cos(\phi) \end{pmatrix}$ be a unit vector orthogonal to $v$. Since $(g - I)^2 = 0$, $gw - w$ is an eigenvector of $g$. Thus $gw - w = \mu v$ for some $\mu \in \mathbb{R}$. This shows that $g$ has above representation.

Hence, for each $n \in \mathbb{N}$ and $\omega \in \Omega$ there exist $\mu_n(\omega) \in \mathbb{R}$ and $\phi_n(\omega) \in [0, \pi]$ such that

$$X_{A_n}^{\bar{\gamma}_n}(\omega, q_n) = R(\phi_n(\omega)) \begin{pmatrix} 1 & \mu_n(\omega) \\ 0 & 1 \end{pmatrix} R(\phi_n(\omega))^{-1} \,,$$

where $R(\phi_n(\omega))$ is the rotation matrix described above. We claim that

$$|\mu_n(\omega)| \leq e^{2||A_0||_r q_n} \,, \quad \text{for each } n \in \mathbb{N} \text{ and } \omega \in \Omega \,. \tag{3.5}$$

This follows at once from inequality (3.1) and the fact $|\mu_n(\omega)| \leq \sqrt{1 + |\mu_n(\omega)|^2} = ||X_{A_n}^{\bar{\gamma}_n}(\omega, q_n) \begin{pmatrix} 0 \\ 1 \end{pmatrix}||$.

Step (4) : Let $C_1$ be a constant such that $||g|| \leq C_1 ||g||_\infty$ for all $g \in sl(2, \mathbb{R})$, where $||\ ||$ and $||\ ||_\infty$ are the uniform (i.e. operator) norm and the supremum norm on the set of matrices respectively, (without loss of generality let $C_1 > 1$). Thus, for any matrix $g \in sl(2, \mathbb{R})$ of the form $g = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$, we have

$$ln \, ||g|| \leq \text{Max} \, \{ln \, (C_1) \, , ln \, (C_1 |\nu|)\} \,.$$

(Note that the if $|\nu| \leq 1$ then $||g||_\infty = 1$, else it is $|\nu|$). In particular, since for any $n \in \mathbb{N}$ the flow $\{T_t^{\gamma_n}\}_{t \in \mathbb{R}}$ is periodic, for any $\ell \in \mathbb{N}$ and $\omega \in \Omega$, we have

$$ln \, ||X_{A_n}^{\bar{\gamma}_n}(\omega, \ell q_n)|| = ln \, ||X_{A_n}^{\bar{\gamma}_n}(\omega, q_n)^\ell|| \leq \text{Max} \, \{ln \, (C_1) \, , ln \, (C_1 |\ell \mu_n(\omega)|)\} \,.$$

Now select $m \in \mathbb{N}$ such that

$$\text{Max} \, \{\frac{ln \, (C_1)}{m} \, , \frac{ln \, (C_1 m)}{m} + \frac{2||A_0||_r}{m}\} < \frac{1}{4k} \,. \tag{3.6}$$

Hence for any $n \in N$ and $\omega \in \Omega$

$$\begin{aligned}
\frac{1}{mq_n} ln\, ||X^{\bar{\gamma}_n}_{A_n}(\omega, mq_n)|| &\leq \text{Max}\,\{\frac{ln\, C_1}{mq_n}, \frac{ln\,(C_1 m|\mu_n(\omega)|)}{mq_n}\} \\
&\leq \text{Max}\,\{\frac{ln\, C_1}{m}, \frac{ln\,(C_1 m|\mu_n(\omega)|)}{mq_n}\} \quad \text{since } q_n \geq 1\,, \\
&\leq \text{Max}\,\{\frac{ln\, C_1}{m}, \frac{ln\,(C_1 m e^{2||A_0||_r}}{m})\,, \quad \text{(by (3.5))} \\
&\leq \text{Max}\,\{\frac{ln\, C_1}{m}, \frac{ln\,(C_1 m)}{m} + \frac{2||A_0||_r}{m}\} \\
&\leq \frac{1}{4k}\,, \quad \text{by (3.6)}\,. \qquad (3.7)
\end{aligned}$$

Step (5) : Now select $n_2 \in N$ such that if $n > n_2$ then

$$C^r \frac{e^{4||A_0||mq_n}}{e^{rq_n^{1+\delta}}} < \frac{1}{4k}\,. \qquad (3.8)$$

Fix any $n \geq Max\{N, n_1, n_2\}$ and consider $A_n$. Clearly, $||A_n - A_0|| < \eta$. We show that $A_n \notin V(k)$. Consider

$$\begin{aligned}
\frac{1}{mq_n} ln\, ||X^{\bar{\gamma}}_{A_n}(\omega, mq_n)|| &= \frac{1}{mq_n}\big(ln\, ||X^{\bar{\gamma}}_{A_n}(\omega, mq_n)|| - ln\, ||X^{\bar{\gamma}_n}_{A_n}(\omega, mq_n)||\big) + \frac{1}{mq_n} ln\, ||X^{\bar{\gamma}_n}_{A_n}(\omega, mq_n)|| \\
&\leq \frac{1}{mq_n}\big(||X^{\bar{\gamma}}_{A_n}(\omega, mq_n) - X^{\bar{\gamma}_n}_{A_n}(\omega, mq_n)||\big) + \frac{1}{4k}\,, \quad \text{by (3.7)}\,. \qquad (3.9)
\end{aligned}$$

Here we have used the fact that (i) $|ln(x) - ln(y)| \leq |x - y|$ if $x, y \geq 1$ and (ii) if $g \in SL(2, \mathbb{R})$ then $||g|| \geq 1$. Now using the estimate (3.4) with $t = mq_n$ we get

$$\frac{1}{mq_n} ||X^{\bar{\gamma}}_{A_n}(\omega, mq_n) - X^{\bar{\gamma}_n}_{A_n}(\omega, mq_n)|| \leq \frac{1}{mq_n} \big|\bar{\gamma} - \bar{\gamma}_n\big|^r mq_n e^{4||A_0||_r mq_n} \leq \frac{C^r e^{4||A_0||_r mq_n}}{e^{rq_n^{1+\delta}}} \leq \frac{1}{4k}\,, \quad \text{by (3.8)}\,.$$

This shows that

$$\frac{1}{mq_n} ln\, ||X^{\bar{\gamma}}_{A_n}(\omega, mq_n)|| < \frac{1}{2k}\,.$$

This means $A_n \notin V(k)$ and the proof is complete. ∎

**Remark 3.1** *From the proof it follows that in the appropriate hypotheses of the main theorem, the 'super Liouville condition' can be replaced by the following weaker condition:*

$$|\bar{\gamma} - \bar{\gamma}_n| \leq Ce^{-f(n)q_n}\,, \quad 1 \leq i \leq d\,,$$

*where $C$ is a constant and $f : \mathbb{N} \to [0, \infty)$ is a function such that $\lim_{n\to\infty} f(n) = \infty$.*

# References


[1] J. Bochi : Genericity of zero Lyapunov exponents, Ergodic Theory and Dynamical Systems 22 (2002), 1667-1696.

[2] J. Bochi and M. Viana : The Lyapunov exponents of generic volume preserving and symplectic maps, Ann. of Math. 161 (2005), 1423-1485.

[3] R. Fabbri : Genericita dell'iperbolicita nei sistemi differenziali lineari di dimensione due, Ph.D. Thesis, Universita di Firenze (1997).

[4] R. Fabbri and R. Johnson : On the Lyapunov exponent of certain $SL(2,\mathbb{R})$ valued cocycles, Differential Equations and Dynamical Systems 7 (1999), 349-370.

[5] R. Fabbri, R. Johnson and R. Pavani : On the nature of the spectrum of the quasi-periodic Schrodinger operator, Nonlinear Analysis : Real World Applications 3 (2002), 37-59.

[6] R. Johnson : Exponential dichotomy, rotation number and linear differential operatorss with bounded coefficients, Jour. Diff. Equn. 61 (1986), 54-78.

[7] R. Johnson and J. Moser : The rotation number for almost periodic potentials, Comm. Math. Phys. 84 (1982), 403-438.

[8] R. Johnson, K, Palmer and G. Sell : Ergodic properties of linear dynamical systems, Siam J. Math. Anal. 18 (1987), 1-33.

[9] J. Moser : An example of a Schrodinger equation with almost periodic potential and nowhere dense spectrum, Comment. Math. Helvetici 56 (1981), 198-224.

[10] M. Nerurkar : Positive exponents for a dense class of continuous $SL(2,\mathbb{R})$ valued cocycles which arise as solutions to strongly accessible linear differential systems, Contemp. Math. 215 (1998), 265-278.

[11] M. Nerurkar : Density of positive Lyapunov exponents in the smooth category, preprint (2008).