On $SL(2,\mathbb{R})$ valued smooth proximal cocycles and cocycles with positive Lyapunov exponents over irrational rotation flows

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Dedicated to Professor Russell Johnson on the occasion of his birthday

Abstract

Consider the class of C^r-smooth $SL(2,\mathbb{R})$ valued cocycles, based on the rotation flow on the two torus with irrational rotation number α. We show that in this class, (i) cocycles with positive Lyapunov exponents are dense and (ii) cocycles that are either uniformly hyperbolic or proximal are generic, if α satisfies the following Liouville type condition: $\left|\alpha - \frac{p_n}{q_n}\right| \le C\exp(-q_n^{r+1+\kappa})$, where $C > 0$ and $0 < \kappa < 1$ are some constants and $\frac{P_n}{q_n}$ is some sequence of irreducible fractions.

Keywords Cocycles, Lyapunov exponents, irrational rotations, proximal extensions, fast periodic approxiamtion

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1 Introduction

This note is regarding a quantitative refinement of a 'rotation number argument' originally due to Russell Johnson (cf. [13]), also see [5],[6], [7]) for $SL(2,\mathbb{R})$ valued cocycles. We shall demonstrate the application of this refined argument by proving (i) the density of smooth cocycles with positive Lyapunov exponents and (ii) genericity of smooth cocycles that are either uniformly hyperbolic or proximal, where the underlying flow is the irrational rotation flow with a 'super Liouvillian' rotation number.

First, we shall discuss a bit of history of such results. In the discrete setting, density of $SL(2,\mathbb{R})$ valued cocycles with positive exponents was first proved by O. Knill in the category of measurable cocycles, (cf. [18], also cf. [8]). Knill's proof uses 'the Herman trick' yielding a lower bound on the exponent. For $\mathbb{R}$ actions (i.e. in the context of cocycles that are fundamental matrix solutions to a linear differential equations), such a density result was proved by R. Johnson in the continuous category, (cf. [12] and also [22]). R. Johnson's argument is based on a result of S. Kotani, (cf. [19]).

For the irrational rotation flow on the 2-torus, with a generic choice of the irrational rotation number, R. Johnson (cf. [12]) and later R. Fabbri and R. Johnson (cf. [6]), proved a density result for cocycles with positive exponents in the continuous and smooth category of cocycles respectively. As mentioned before, these proofs are based on the work of Kotani along with certain properties of 'the rotation number' of a cocycle. It is this result we want to refine by fixing the rotation number of the base flow and imposing a certain 'Liouville type' condition on it.

Regarding proximality, generic proximal behaviour was established (using the so called 'conjugacy approximation tehnique') in the very restrictive class of 'closures of smooth coboundaries', (see [21]). An existence result using the same technique but in the continuous category was proved in ([10]). However the class of 'closures of coboundaries' is too special and rather unnatural in the context of cocycles arising as the fundamental matrix solutions of linear differential systems. In the class of all cocycles proximal cocycles are not even dense, because the uniformly hyperbolic cocycles are never proximal and uniformly hyperbolic cocycles form an open set. Thus our result is the first such result and in fact it has the flavour similar to 'Mane's cojucture' - a linearized version of which was proved by J. Bochi (see [2]). The result of Bochi says (in the discrete setting) that a generic continous cocycle is either uniformly hyperbolic or has zero exponents. Our result is a kind of 'topological dynamic' analogue where the condition of zero exponents is replaced by proximality. However, there are certain differences between the condition of proximality and that of zero exponents. For example, our proximality result is in the class of C^r, ($r \in \mathbb{N}$), cocycles but it is not known whether J. Bochi's result is true in the smooth (even C^1) category. We mention that in a recent joint work with R. Johnson ([15]) we proved such a result in the class of Hölder cocycles. Comparing techniques, the one used for the 'proximality result' needs a suitable perturbation only on a piece of a single orbit, whereas for the 'zero exponents result' the desired perturbation has to have a (measure theoretically) large support. On the other hand, for obtaining proximality and positive exponents by small perturbation, first one gets a suitable perturbation by the 'rotation number argument' and then needs to modify it further using quantitative perturbation results, (such as Propositions (2.6) and (3.1)) and an 'open mapping type result', (Proposition (2.7)). For obtaining zero exponents the perturbation obtained by the 'rotation number argument' suffices and does not need any additional modifications, ([15]).

In passing we also mention other more recent results regarding density of cocycles with positive exponents in the smooth category. These results are in the discrete setting and are based on techniques completely different than those mentioned above. First, the work of R. Krikorian on reducibility of $SL(2,\mathbb{R})$ valued cocycles, (see [17]), yields such a result when the base transformation is a circle rotation where the rotation number satisfies a stronger than Diophantine condition, (a 'recurrent diophantine condition'). More recently this result is generalized in [9]. A recent result of A. Avila, ([1]), proves density of smooth $SL(2,\mathbb{R})$ valued cocycles with positive exponents without any assumption on the base rotation number other than irrationality. This is based on refining, (the discrete version of), S. Kotani's theorem. Yet another recent result of M. Viana establishes density in the smooth category when the base transformation is an Anosov diffeomorphism, ([23]). As mentioned above, all these results are in the discrete setting where one deals with the space of cocycles directly, (in contrast our method deals with the space of their 'infinitisimal generators').

Now we comment on our approach and technique. As mentioned in the begining, we develop a quantitative version of R. Johnson's argument and use S. Kotani's result to reduce the problem to constructing smooth unbounded cocycles. To show that smooth unbounded cocycles are generic, we need to develop two key quantitative results (Proposition (2.6) and Proposition (2.7)). The first of these is a quantitative statement about perturbing a parabolic matrix and the second is a 'quantitative open mapping theorem'. For the proximality result, we need to develop another suitable quantitative result, (Proposition (3.1)). For simplicity the paper is written for the irrational rotation flow on the two torus, but the arguments can be generalized to more general flows. The extension to irrational rotation flows on n torus, (with appropriate super Liouville condition), is more or less direct and one can abstract our method to extend these results to flows on manifolds admiting 'fast periodic approximation' and having some additional 'geometric structure'.

Before we formally begin by introducing basic definitions, I wish to dedicate this paper to Professor Russell Johnson in celebrating his 60^{th}. I truely cherish our friendship over the past twenty five years or more. It was his early work that introduced me to applying abstract theory of dynamical systems to the qualitative theory of non-autonomous linear systems.

Definition 1.1 *A flow* $(\Omega, \{T_t\}_{t\in\mathbb{R}})$ *consists of a compact metric space* Ω *and a one parameter group* $\{T_t\}_{t\in\mathbb{R}}$ *of homeomorphisms of* Ω *such that the action* $(\omega, t) \to T_t(\omega) \in \Omega$ *is jointly continuous. If* Ω *is a* C^∞ *manifold and the action is jointly* C^k, $(k \in \mathbb{N})$, *then the flow is said to be a* C^k *flow.*

Example 1.2 *Our prime example of a flow is the rotation flow on the 2-torus-*$\mathcal{T}^2$*. We shall think of* $\Omega = \mathcal{T}^2$ *as the unit square* $[0,1]\times[0,1]$ *whose opposite end points are identified. The flow on* Ω *is given by the rule*

$$T_t^\gamma(\omega) = T_t^\gamma(x,y) = (x+\gamma t, y+t), \quad \omega = (x,y) \in [0,1]\times[0,1]\,, t \in \mathbb{R}\,, \tag{1.1}$$

where $\gamma \in \mathbb{R}$ *is the rotation number of the flow.*

First we shall recall the basic definitions and facts regarding a general C^r flow $(\Omega, \{T_t\}_{t\in\mathbb{R}})$. Let $A : \Omega :\to gl(2,\mathbb{R})$ be a given C^r function. Consider the family of linear differential equations parametrized by points of Ω:

$$x' = A(T_t\omega)x, \qquad x \in \mathbb{R}^2,\ \omega \in \Omega\,. \tag{1.2}$$

For each $\omega \in \Omega$, let $t \to X_A(\omega, t)$ be the fundamental matrix solution of the equation (1.2) satisfying $X_A(\omega, 0) = I$-the 2×2 identity matrix. Then the map $X_A : \Omega\times\mathbb{R} \to GL(2,\mathbb{R})$ is a cocycle, i.e. it is continuous and satisfies the following cocycle identity:

$$X_A(\omega, t+s) = X_A(T_t(\omega), s)X_A(\omega, t) \qquad \text{for all } \omega \in \Omega,\ t, s \in \mathbb{R}\,. \tag{1.3}$$

Note that if $A : \Omega :\to sl(2,\mathbb{R})$, then $X_A : \Omega\times\mathbb{R} \to SL(2,\mathbb{R})$.

Let $\mathbb{P}$ denote the projective 1-space. A cocycle X_A defines a skew product flow on the projective bundle $\mathbb{P}\times\Omega$ given by the action

$$T_t^A([v]\,, \omega) = ([X_A(\omega,t)v]\,, T_t(\omega)), \quad ([v]\,,\omega) \in \mathbb{P}\times\Omega\,, t\in\mathbb{R}\,, \tag{1.4}$$

where $[v]$ denotes the ray through a non-zero vector $v \in \mathbb{R}^2$. Thus, the flow $(\mathbb{P}\times\Omega, \{T_t^A\}_{t\in\mathbb{R}})$ defines a dynamical extension of the flow $(\Omega\,, \{T_t\}_{t\in\mathbb{R}})$ via the factor map $\pi([v],\omega) = \omega$.

Definition 1.3 *A cocycle* X_A *is proximal if the factor map* π *defines a proximal extension. This means that for any* $\omega \in \Omega$, *if* $([v_1],\omega)\,,([v_2]\,,\omega))$ *are any two distinct points in* $\mathbb{P}\times\Omega$, *in the fiber over* ω, *then there exists a sequence* $t_n \in \mathbb{R}$ *such that* $|t_n| \to \infty$ *and* $d\big(T_{t_n}^A([v_1]\,,\omega)\,, T_{t_n}^A([v_2]\,,\omega)\big) \to 0$, *where* d *is any metric generating the product topology on* $\mathbb{P}\times\Omega$.

Lyapunov exponents and Exponential dichotomy

Next, we recall the definition of the (largest) Lyapunov exponent of a cocycle.

Definition 1.4 *Let* $X_A : \Omega\times\mathbb{R} \to GL(2,\mathbb{R})$ *be a cocycle. Let*

$$\beta(X_A) = \lim_{t\to\infty} \frac{1}{t} ln||X_A(\omega,t)||\,.$$

Given an invariant measure on the flow $(\Omega, \{T_t\}_{t\in\mathbb{R}})$, *the Oseledets multiplicative ergodic theorem guarantees the existence of this limit a.e.* ω *with respect to that measure on* Ω.

Another way to capture the logarithmic growth rate of the solutions is through the notion of uniform hyperbolicity (or exponential dichotomy).

Definition 1.5 *A cocycle $X_A : \Omega \times \mathbb{R} \to GL(2,\mathbb{R})$ is said to have an exponential dichotomy (ED) if there are constants $K\,,\rho > 0$ and a continuous projection valued map $\hat{P} : \Omega \to Proj(\mathbb{R}^2) : \omega \to \hat{P}_\omega$ such that*

$$||X_A(\omega,t)\hat{P}_\omega X_A(\omega,s)^{-1}|| \le Ke^{-\rho(t-s)} \quad if\, t \ge s\,, \tag{1.5}$$
$$||X_A(\omega,t)(I-\hat{P}_\omega)X_A(\omega,s)^{-1}|| \le Ke^{\rho(t-s)} \quad if\, t \le s\,. \tag{1.6}$$

The set $\Sigma(A)$ defined by

$$\Sigma(A) = \{\lambda \in \mathbb{R} \mid \textit{the cocycle } e^{-\lambda t}X_A(\omega,t) \textit{ does not admit an ED}\},$$

is called the dichotomy spectrum of the cocycle X_A.

For a cocycle $X_A : \Omega \times \mathbb{R} \to SL(2,\mathbb{R})$ when the base flow is the rotation flow with irrational winding vector, (or more generally, any uniquely ergodic flow), there are exactly three possibilities described in the following proposition (cf. [16]).

Proposition 1.6 *For any minimal uniquely ergodic flow, given a cocycle X_A into $SL(2,\mathbb{R})$, the dichotomy spectrum $\Sigma(A)$ is either a singleton set $\{0\}$ or a two point set $\{-\beta(X_A)\,,\beta(X_A)\}$ or the interval $[-\beta(X_A)\,,\beta(X_A)]$.*

The first case is equivalent to saying $\beta(X_A) = 0$-which we shall refer to as the zero exponent case. In the second case the cocycle X_A is said to be uniformly hyperbolic. It is exactly the case when X_A admits an exponential dichotomy. We also remark that a cocycle cannot be both, proximal and uniformly hyperbolic. In the third case the cocycle is said to be non-uniformly hyperbolic.

In the rest of the paper, whenever our base flow is the rotation flow on the 2-torus with winding number γ and $A : \Omega \to sl(2,\mathbb{R})$ is a given map, the cocycle X_A generated by A will be denoted by X_A^γ. The dependance of the cocycle on γ will be very crucial in our arguments. Now we state our results.

Theorem 1.7 *Let $(\Omega, \{T_t\}_{t\in\mathbb{R}})$ be the irrational rotation flow on the 2-torus with winding number α. Let $r \in \mathbb{N}$. Suppose α satisfies the following super Liouvillian condition :*

$$|\alpha - \alpha_n| \le Ce^{-q_n^{r+1+\kappa}}\,, \quad n \in \mathbb{N}\,, \tag{1.7}$$

where $C > 0$ and $0 < \kappa < 1$ are some constants and $\alpha_n = \frac{p_n}{q_n} \in \mathbb{Q}$, where $p_n\,, q_n \in \mathbb{N}$, $(p_n, q_n) = 1$, and $q_n \to \infty$. Fix any $\omega^* \in \Omega$. Let

$$C^r_{unb} = \{A \in C^r(\Omega, sl(2,\mathbb{R})) \mid \textit{the set } \{X_A^\alpha(\omega^*,t) \mid t > 0\} \textit{ is unbounded in } SL(2,\mathbb{R})\}\,.$$

Then C^r_{unb} is residual (in particular dense) in $C^r(\Omega, sl(2,\mathbb{R}))$.

Theorem 1.8 *Let $(\Omega, \{T_t\}_{t\in\mathbb{R}})$ be as in the above theorem. Then the set*

$$C^r_{pos} = \{A \in C^r(\Omega, sl(2,\mathbb{R})) \mid \beta(X_A^\alpha) > 0\}$$

is dense in $C^r(\Omega, sl(2,\mathbb{R}))$.

Theorem 1.9 *Let* $(\Omega, \{T_t\}_{t\in\mathbb{R}})$ *be as in the above theorem. Then the set*

$$C^r_{prox,uh} = \{A \in C^r(\Omega, sl(2,\mathbb{R})) \mid X^\alpha_A \text{ is either uniformly hyperbolic or proximal}\}\,,$$

is residual in $C^r(\Omega, sl(2,\mathbb{R}))$.

First we show that the proof of Theorem (1.8) can be reduced to that of Theorem (1.7) by R. Johnson's argument which is based on a generalization of a theorem of S. Kotani. To discuss Kotani's theorem, consider a linear differential system based on a general uniquely ergodic flow $(\Omega, \{T_t\}_{t\in\mathbb{R}})$. One introduces a 'spectral parameter' $\lambda \in \mathbb{C}$ into the linear system by considering

$$x' = [A(T_t\omega) + \lambda J]x, \qquad x \in \mathbb{R}^n,\ \omega \in \Omega\,, \tag{1.8}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $X_{A+\lambda J}$ denote the cocycle generated by the above differential system. The largest Lyapunov exponent $\beta(\lambda) \equiv \beta(X_{A+\lambda J})$ of the system (1.8) is a function of λ. The following is a generalization of a result of S. Kotani (cf. [3]).

Theorem 1.10 *Consider the system (1.8) with* $A \in C(\Omega, sl(2,\mathbb{R}))$*, based on a minimal uniquely ergodic flow* $(\Omega, \{T_t\}_{t\in\mathbb{R}}, \mu)$*. Let* I *be an open interval containing* 0*. Suppose* $\beta(\lambda) \equiv \beta(X_{A+\lambda J}) = 0$ *for almost all* $\lambda \in I$ *with respect to the Lebesgue measure. Then there exists a compact set* $K \subset SL(2,\mathbb{R})$ *such that*

$$X_A(\omega, t) \in K\,, \quad \text{for all } t \in \mathbb{R} \quad \text{and for all } \omega \in \Omega\,.$$

Proof of Theorem (1.8)

Proof: Fix a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of $SL(2,\mathbb{R})$ such that $K_n \subset (K_{n+1})^0$, for $n \in \mathbb{N}$ and $SL(2,\mathbb{R}) = \cup_{n\in\mathbb{N}} K_n$. For contradiction, suppose there is an open set $V \subset C^r(\Omega, sl(2,\mathbb{R}))$ such that for each $B \in V$, the Lyapunov exponent $\beta(X^\alpha_B) = 0$. Let $B_0 \in V$ and consider the family of equations

$$x' = [B_0(T^\alpha_t(\omega) + \lambda J]x\,, \quad x \in \mathbb{R}^2\,. \tag{1.9}$$

Since B_0 is in the interior of V, there exists an interval $I \subset \mathbb{R}$ containing 0 such that for $\lambda \in I$ the C^r function $B_0 + \lambda J$ belongs to V. Hence,

$$\beta(\lambda) \equiv \beta(X^\alpha_{B_0+\lambda J}) = 0 \quad \text{for all } \lambda \in I\,.$$

Hence by Theorem (1.10)

$$X^\alpha_{B_0}(\omega, t) \in K_m\,, \quad \text{for all } (\omega, t) \in \Omega \times \mathbb{R}\,,$$

for some $m \in \mathbb{N}$.

Next, recall that $\omega^* \in \Omega$ is some fixed point. For $n \in \mathbb{N}$, set

$$W_n = \{B \in V \subseteq C^r(\Omega, sl(2,\mathbb{R})) \mid X^\alpha_B(\omega^*, t) \in K_n,\ \text{for all } t \in \mathbb{R}\}\,.$$

Since B_0 in the above argument was an arbitrary point in V, it follows that

$$V \subseteq \cup_{n\in\mathbb{N}} W_n\,.$$

Since each W_n is a closed subset of the Baire space $C^r(\Omega, sl(2,\mathbb{R}))$, there is some $n_0 \in \mathbb{N}$ for which the set W_{n_0} contains a non-empty open subset of V. This contradicts the density of the set $C^r_{unb}(\Omega, sl(2,\mathbb{R}))$ of unbounded cocycles guaranteed by Theorem (1.7). ∎

Now we proceed to the proofs of our main theorems, Theorem (1.7) and Theorem (1.9).

2 Proof of Theorem (1.7)

If g is 2×2 matrix, $||g||$ will denote its uniform (i.e. operator) norm, (where the underlying vector space $\mathbb{R}^2$ carries the standard Euclidean norm). For convinience, we shall take ω^* to be $\omega^* = (0,0) \in \Omega$, the proof does not depend on which ω^* is selected. For each $N \in \mathbb{N}$, set

$$E(N) = \{A \in C^r(\Omega, sl(2,\mathbb{R})) \mid ||X_A^\alpha(\omega^*, t)|| > N \text{ for some } t > 0\}.$$

Clearly $E(N)$ is open in $C^r(\Omega, sl(2,\mathbb{R}))$, this is a consequence of continuous dependence of solutions on the vector field. More precisely, it follows from the continuity of the map $A \to X_A^\alpha(\omega^*, r)$: $C^0(\Omega, sl(2,\mathbb{R})) \to SL(2,\mathbb{R})$. We shall show that $E(N)$ is also dense in $C^r(\Omega, sl(2,\mathbb{R}))$. The proof then follows from the Baire Category Theorem by considering the set $\cap_{N\in\mathbb{N}} E(N)$.

Suppose $E(N)$ is not dense in $C^r(\Omega, sl(2,\mathbb{R}))$. This means there exists

(1) a function $A_0 \in C^r(\Omega, sl(2,\mathbb{R}))$ and

(2) an $\varepsilon_0 > 0$ such that

(3) if $||A - A_0||_r < \varepsilon_0$ then $||X_A^\alpha(\omega^*, t)|| \le N$ for all $t \in [0, \infty)$.

We shall construct a function $B \in C^r(\Omega, sl(2,\mathbb{R}))$ such that
(i) $||B - A_0||_r < \varepsilon_0$, and
(ii) $B \in E(N)$, that is, there is some $t > 0$ such that $||X_B^\alpha(\omega^*, t)|| > N$. This will contradict the hypothesis and prove the density of $E(N)$.

Remark 2.1 (A general comment about the proof) *Construction of such a function B in the class $C^0(\Omega, sl(2,\mathbb{R}))$ of continuous cocycles is much easier (cf. [12] and [22]). To construct such a function in the smooth category, following R. Johnson, we need to first employ certain arguments involving the 'rotation number' of the cocycle (cf. [13], [6] and [7]). Once this is done, our quantitative refinement of R. Johnson's argument begins. This refinement is based on two key results, Lemma (2.6) and Proposition (2.7) and then followed by an application of the Gronwall's inequality, (Lemma (2.8)).*

Thus proofs of both Theorem (1.7) and Theorem (1.9) have four ingradients : (A) A rotation number argument, (B) a quantitative result about perturbation of a parabolic matrix, (C) a quantitative open mapping theorem and (D) Gronwall's inequality. For Theorem (1.9) we need to suitably modify 'ingradients' (B) and (D) but the overall strategy is the same. The key observation common to both proofs is the following : Suppose A is a map such that no map close enough to A generates a uniformly hyperbolic cocycle. Then one can find an arbitrarily small perturbation B of A so that $X_B^\alpha(\omega^*, t)$ can be approximated by parabolic matrices, for large values of t, of the form $t = mq_n$, where q_n is the n^{th} convergent of α and m is an appropriate positive integer which admits a crucial 'apriory upper estimate'.

We begin by describing the 'rotation number argument', followed by precise statements of the two key results and a version of Gronwall inequality. The construction of the function B can only begin after these preliminaries because choices of various constants to be made in this construction depend on various parameters appearing in the statements of the two key propositions.

A : The rotation number argument

In this section we give a brief introduction of the rotation number of a cocycle, state its basic properties and describe the 'rotation number argument' which proves Proposition (2.5). This proposition plays a crucial role in our proof. The flow $(\Omega, \{T_t^\gamma\}_{t\in\mathbb{R}})$ is the rotation flow on the d torus with rotation number γ. Consider the family

$$x' = A(T_t^\gamma(\omega))x\,, \quad \omega\in\Omega\,, \quad t\in\mathbb{R}\,,$$

where $A = \begin{pmatrix} d & -b+c \\ b+c & -d \end{pmatrix}$ and b, c, d are continuous real valued functions of ω. Introducing the usual polar co-ordinates (r, θ), the above linear equation can be written as

$$\begin{aligned} r' &= \big[d(T_t^\gamma\omega)cos(2\theta) + c(T_t^\gamma\omega)sin(2\theta)\big]r \\ \theta' &= b(T_t^\gamma\omega) + c(T_t^\gamma\omega)cos(2\theta) - d(T_t^\gamma\omega)sin(2\theta) \end{aligned}$$

Note that the θ equation does not depend on r.

Definition 2.2 *The rotation number $\rho \equiv \rho(\omega, \gamma, A)$ of the above family is defined by setting*

$$\rho = \lim_{t\to\infty} \frac{\theta(t)}{t}\,,$$

where $\theta(t)$ is a solution of the θ equation with arbitrary initial condition $\theta(0) = \theta_0$.

We list the basic properties of α, (cf. [11] and [14]) for details).

Rotation Number : Continuity Properties

(**1**) The above limit exists and is independent of the initial condition θ_0. Furthermore, if the flow is minimal it is independent of ω.
(**2**) For each fixed $\gamma\in\mathbb{R}$ the map $(\omega, A) \to \rho(\omega, \gamma, A)$ continuous.
(**3**) If $\gamma\notin\mathbb{Q}$ and if $\gamma_n\in\mathbb{R}$ is such that $\gamma_n\to\gamma$, then $(\omega, \gamma_n, A) \to \rho(\omega, \gamma, A)$ where the convergence is uniform on compact subsets of $\Omega\times C^0(\Omega, sl(2,\mathbb{R}))$.

Rotation Number : Spectral Properties

The dichotomy property is closely related to the spectral properties of certain differential operators. This relation results with an introduction of a 'spectral parameter' λ. This is done as follows: given $A\in C^0(\Omega, sl(2,\mathbb{R}))$, consider the system

$$x' = [A(T_t^\gamma\omega) + \lambda J]x\,, \tag{2.1}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\lambda\in\mathbb{R}$ or more generally in $\mathbb{C}$. The system generates the cocycle $X^\gamma_{A+\lambda J}$. Also consider the following AKNS operator L_ω associated with A,

$$L^A_\omega = J^{-1}\big[\frac{d}{dt} - A(T_t^\gamma\omega)\big]\,.$$

This is viewed as an unbounded self-adjoint operator on the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2)$ of square integrable $\mathbb{C}^2$-valued functions on $\mathbb{R}$.

(**1**) The following proposition describes the resolvent of this operator in terms of the notion of ED, (for details and proofs cf. [11] as well as [3]).

Proposition 2.3 *(a) Let $A\in C^0(\Omega, sl(2,\mathbb{R}))$. A complex number λ is in the resolvent of L^A_ω if and only if the cocycle $X^\gamma_{A+\lambda J}$ admits ED.*
(b) Suppose that the cocycle $X^\gamma_{A+\lambda J}$ generated by the system (2.1) does not admit ED for all $\lambda\in I$, where $I\subset\mathbb{R}$ is some open interval containing 0*. Then the map $\lambda\to\rho(\omega, \gamma, A+\lambda J)$ is strictly increasing.*

(**2**) We shall also need the following proposition. This is based on the fact : Given a cocycle X_A based on a periodic base flow with period T, the polar component of the eigenvalue of the 'monodromy matrix' $X_A(\omega^*, T)$ is $e^{i\rho T}$, where ρ is the rotation number of X_A, (see cf. [20] as well as [6]).

Proposition 2.4 *Suppose the flow $(\Omega, \{T_t^\gamma\}_{t\in\mathbb{R}})$ is periodic with period $T > 0$. Let $A \in C^0(\Omega, sl(2,\mathbb{R}))$. Suppose for some $\omega \in \Omega$ and $\lambda \in \mathbb{R}$, $\rho(\omega, \gamma, A + \lambda J) = \frac{2\pi m}{T}$ for some integer m. Then λ is either (i) in the closure of a resolvent interval or (ii) λ is a 'closed gap'. In this last case, $tr(X^\gamma_{A+\lambda J}(\omega, T)) = 2$ because the rotation number is an even multiple of $\frac{\pi}{T}$.*

We say that $\lambda \in \mathbb{R}$ is a closed gap for the operator $L_\omega = J^{-1}\left[\frac{d}{dt} - A(T_t^\gamma\omega)\right]$ if the rotation number $\rho(\omega, \gamma, A + \lambda J) = \frac{\pi k}{T}$ for some integer k and if λ is not an end point of a spectral gap of the spectrum of L_ω. Now we shall state the crucial result we need, (see [15], Proposition 4, for complete details).

Proposition 2.5 *(A perturbation lemma based on the rotation number argument) Fix $\omega^* \in \Omega$. Let $\gamma \notin \mathbb{Q}$. Let $A_0 \in C^r(\Omega, sl(2,\mathbb{R}))$. Suppose that for some $\varepsilon > 0$, the cocycle X_A^γ does not admit ED for any A such that $||A - A_0||_r < \varepsilon$. Fix a sequence $\gamma_n \to \gamma$ so that each flow $(\Omega, \{T_t^{\gamma_n}\}_{t\in\mathbb{R}})$ is periodic with period q_n and $q_n \to \infty$.*

Then given any $\xi > 0$, there exists some $n_1 \in \mathbb{N}$ such that for each $n \geq n_1$ there exists a function $A_n \in C^r(\Omega, sl(2,\mathbb{R}))$ such that

(1) $||A - A_n||_r < \xi$ and

(2) $tr\big(X^{\gamma_n}_{A_n}(\omega^*, q_n)\big) = 2$.

B : Quantitative perturbation lemma I

For the proof of both main theorems, we need quantitative statements about perturbing a parabolic matrix. The following lemma is the first one, the second will be proved in the next section. First we remark that we need to consider both, the uniform norm $||\,||$ and the supremum norm $||\,||_\infty$ on the vector space of 2×2 real matrices. Let C_1 and C_2 be constants such that $||g|| \leq C_1||g||_\infty$ and $||g||_\infty \leq C_2||g||$ for all 2×2 matrices g. Let P denote the set of parabolic matrices, i.e.

$$P = \{g \in SL(2,\mathbb{R}) \mid tr(g) = 2\}.$$

Lemma 2.6 *(A quantitative perturbation lemma) Let $M > C_2$ be given. Let $P(M)$ be the compact set of parabolic matrices given by*

$$P(M) = \{g \in P \mid ||g|| \leq M\}.$$

Then there exists $\eta_0 \equiv \eta_0(M) > 0$ and a constant $L \equiv L(M) > 0$ such that the following holds:

For any $\eta \in (0, \eta_0)$ there exists $m \equiv m(M, \eta) \in \mathbb{N}$ such that

(1) $m \leq \frac{L}{\eta}$ and

(2) given any $g \in P(M)$, there exists $g_1 \in SL(2,\mathbb{R})$ such that

(2_a) $||g - g_1|| < \eta$, and

(2_b) $||g_1^m|| > M$.

Proof: Note that if $g \in SL(2,\mathbb{R})$ and $tr(g) = 2$, then g can be written as

$$g \equiv g_{\mu,\varphi} = R(\varphi) \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} R(\varphi)^{-1} ,$$

for some $\varphi \in [0, \pi)$ and $\mu \in \mathbb{R}$, where $R(\varphi) = \begin{pmatrix} cos\varphi & sin\varphi \\ -sin\varphi & cos\varphi \end{pmatrix}$. To see this, note that the characteristic equation of g is $\lambda^2 - 2\lambda + 1 = 0$ and g has a unique eigenvalue 1 with algebraic multiplicity 2. Let $v = \begin{pmatrix} cos(\varphi) \\ sin(\varphi) \end{pmatrix}$ be an eigenvector of g of standard Euclidean norm 1, and let $w = \begin{pmatrix} -sin(\varphi) \\ cos(\varphi) \end{pmatrix}$ be the unit vector orthogonal to v. Since $(g - I)^2 = 0$, $gw - w$ is an eigenvector of g. Thus $gw - w = \mu v$ for some $\mu \in \mathbb{R}$. This shows that parabolic g has such a parametric representation $g \equiv g_{\mu,\varphi}$, $\varphi \in [0, \pi)$ and $\mu \in \mathbb{R}$.

Next, note that

$$||g_{\mu,\varphi} - g_{\mu',\varphi}|| \leq || \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & \mu' \\ 0 & 1 \end{pmatrix} || \leq C_1 || \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & \mu' \\ 0 & 1 \end{pmatrix} ||_\infty \leq C_1 |\mu - \mu'| .$$

Let $\eta > 0$ be given. Then given any $g \equiv g_{\mu,\varphi} \in P(M)$, taking $g' = g_{\mu',\varphi}$, where μ' is chosen so that (i) $|\mu - \mu'| < \frac{\eta}{C_1}$ and (ii) $|\mu'| > \frac{\eta}{2C_1}$. We have made sure that $||g - g'|| < \eta$. Now consider,

$$||(g')^m|| = || \begin{pmatrix} 1 & m\mu' \\ 0 & 1 \end{pmatrix} || \geq C_2 || \begin{pmatrix} 1 & m\mu' \\ 0 & 1 \end{pmatrix} ||_\infty = C_2 m|\mu'| > m\frac{C_2\eta}{2C_1} > M ,$$

where $m \in N$ is chosen be the first positive integer such that $m\frac{\eta}{2C_1} > \frac{M}{C_2} > 1$, (note that our choice has made $m|\mu'| > 1$, which allowed us to write the norm of the last matrix on the above line as $m|\mu'|$). Now we need to only verify the estimate on m. Since $m \leq \frac{2C_1M}{C_2\eta} + 1$, if $\eta_0 > 0$ is chosen so that $1 < \frac{C_1M}{C_2\eta_0}$, then for any $\eta \in (0, \eta_0)$, we have

$$m \leq \frac{2C_1M}{C_2\eta} + \frac{C_1M}{C_2\eta} = \frac{L}{\eta} ,$$

where $L = \frac{3C_1M}{C_2}$. ■

C : A quantitative open mapping result

The second key proposition is a quantitative statement about the openness of the 'evaluation map'

$$A \to X_A^{\gamma_n}(\omega^*, q_n) : C^r(\Omega, sl(2,\mathbb{R})) \to SL(2,\mathbb{R}), ,$$

where $\gamma_n \in \mathbb{Q}$ and $\omega^* \in \Omega$ are fixed. It says that given a rational $\gamma_n = \frac{p_n}{q_n} \in [0,1]$ and a $A \in C^r(\Omega, sl(2,\mathbb{R}))$, the above map is open at $A \in C^r(\Omega, sl(2,\mathbb{R}))$ and in fact the image of an ε ball under this map contains a ball of radius $\frac{K}{q_n^r}$, where K is a constant that depends on ε, A and r but is independent of q_n. We also need a certain 'uniformity' with respect to the function A as well as with respect to the rotation number $\frac{p_n}{q_n}$. The precise statement follows and will be proved later, in the last section of the paper.

Proposition 2.7 *(A quantitative open mapping result) Let $M > 0$ and let*

$$\mathcal{F} \subset \mathcal{F}_M \equiv \{A \in C^r(\Omega, sl(2,\mathbb{R})) \mid ||A||_r \leq M\} ,$$

be a family of maps. Fix some closed non-degenerate interval $J \subset (0,1)$ and fix a left invariant metric d^* on $SL(2,\mathbb{R})$. Let $\varepsilon > 0$ be given.

(I) Then there exists a constant $K \equiv K(M,r,\varepsilon,J) > 0$, with the following property: Let $A \in \mathcal{F}$ and $\gamma_n = \frac{p_n}{q_n} \in J$ be any rational. Let $\eta = \frac{K}{q_n^r}$. Then given any $g^* \in SL(2,\mathbb{R})$ such that $d^*(X_A^{\gamma_n}(\omega^*, q_n), g^*) < \eta$, there exist a $B \in C^r(\Omega, sl(2,\mathbb{R}))$ such that
(a) $||B - A||_r < \varepsilon$ and
(b) $X_B^{\gamma_n}(\omega^*, q_n) = g^*$.

In order to apply the above result to our situation we need to replace the left invariant metric by the operator norm on matrices in $SL(2,\mathbb{R})$ and this requires an additional 'boundedness' assumption. We have,
(II) Let $J' \subset J \cap \mathbb{Q}$. If in addition there exists a constant $M_1 > 0$ such that $||X_A^{\gamma_n}(\omega^*, q_n)|| \le M_1$ for all $A \in \mathcal{F}$ and all $\gamma_n \in J'$, then (for any $\gamma_n \in J'$), there exists a constant $K \equiv K(M_1, M, r, \varepsilon, J) > 0$ such that the condition $d^*(X_A^{\gamma_n}(\omega^*, q_n)\,, g^*) < \eta$ can be replaced by $||X_A^{\gamma_n}(\omega^*, q_n) - g^*|| < \eta$, in the conclusion of part (I).

Along with the above two propositions, we shall need the following version of Gronwall's inequality.

D : Gronwall's inequality

Lemma 2.8 *(Gronwall's estimate) Let $A \in C^r(\Omega, sl(2,\mathbb{R}))$ and $\alpha\,, \gamma \in [0,1]$. Suppose that $||X_A^\alpha(\omega^*, t)|| \le M$ for all $t \in [0,\infty)$. Then given any real number $T > 0$, we have*

$$||X_A^\alpha(\omega^*, t) - X_A^\gamma(\omega^*, t)|| \le \frac{MT^2||A||_1}{2}\big|\alpha - \gamma\big|e^{t||A||_0}\,, \quad \text{for all } t \in [0,T]\,, \tag{2.2}$$

Proof: Fix any vector $v \in \mathbb{R}^n$ with $||v|| \le 1$. Let $x^\alpha(t)$ and $x^\gamma(t)$ denote the solutions of the differential equations $x' = A(T_t^\alpha \omega^*)x$ and $x' = A(T_t^\gamma \omega^*)x$ respectively, satisfying the initial condition $x^\alpha(0) = x^\gamma(0) = v$. Then,

$$x^\alpha(t) - x^\gamma(t) = \int_0^t A(T_s^\alpha \omega^*)x^\alpha(s) - A(T_s^\gamma \omega^*)x^\gamma(s)ds$$

yields,

$$||x^\alpha(t) - x^\gamma(t)|| \le \int_0^t ||A(T_s^\alpha \omega^*) - A(T_s^\gamma \omega^*)||\,||x^\alpha(s)||ds + \int_0^t ||A(T_s^\gamma \omega^*)||\,||x^\alpha(s) - x^\gamma(s)||ds\,.$$

Now note that for $s \in [0,t]$, by the mean value theorem

$$\begin{aligned} ||A(T_s^\alpha \omega^*) - A(T_s^\gamma \omega^*)|| &= ||A(x^* + \alpha s, y^* + s) - A(x^* + \gamma s, y^* + s)|| \\ &\le ||\frac{\partial A}{\partial x}||_0\, \big|\alpha - \gamma\big|s \le ||A||_1\big|\alpha - \gamma\big|s\,, \end{aligned}$$

where $\omega^* = (x^*, y^*) \in \mathcal{T}^2$. Furthermore, $||x^\alpha(s)|| = ||X^\alpha(\omega^*, s)v|| \le M||v|| \le M$, for all $s > 0$, by the hypothesis. Thus,

$$||x^\alpha(t) - x^\gamma(t)|| \le \frac{M||A||_1}{2}\big|\alpha - \gamma\big|T^2 + \int_0^t ||A||_0\,||x^\alpha(s) - x^\gamma(s)||ds\,, \quad \text{for } 0 \le t \le T\,.$$

Thus, the result follows by applying the Gronwall's inequality and taking the supremum over all vectors v with $||v|| \le 1$. ■

Proof of the density of E(N)

Suppose $E(N)$ is not dense in $C^r(\Omega, sl(2,\mathbb{R}))$. Thus there exists

(1) a function $A_0 \in C^r(\Omega, sl(2,\mathbb{R}))$ and

(2) $\varepsilon_0 > 0$ such that if $||A - A_0||_r < \varepsilon_0$ then

$$||X_A^\alpha(\omega^*, t)|| \le N\,, \quad \text{for all } \ 0 \le t\,. \tag{2.3}$$

Without loss of generality, we shall assume that

$$\varepsilon_0 < ||A_0||_r\,. \tag{2.4}$$

We shall construct a function $B \in C^r(\Omega, sl(2,\mathbb{R}))$ such that
(i) $||B - A_0||_r < \varepsilon_0$, and
(ii) $B \in E(N)$, i.e. that there is some $t > 0$ such that $||X_B^\alpha(\omega^*, t)|| > N$. This contradicts the assumption and thus density of $E(N)$ will be proved. The construction of map B will be carried out in several steps.

Step (1) (The rotation number argument) : For each A such that $||A - A_0||_r < \varepsilon_0$, the cocycle X_A^α is bounded, hence it does not admit exponential dichotomy. Applying Proposition (2.5) to A_0 with $\gamma = \alpha$, $\gamma_n = \alpha_n \equiv \frac{p_n}{q_n}$, $\varepsilon = \frac{\varepsilon_0}{2}$ and $\xi = \frac{\varepsilon_0}{2}$, we get an $n_1 \in \mathbb{N}$ and an infinite sequence $A_n \in C^r(\Omega, sl(2,\mathbb{R}))$, $n \ge n_1$, such that
(i) $||A_0 - A_n||_r < \frac{\varepsilon_0}{2}$, for all $n \ge n_1$ and
(ii) $tr\big(X_{A_n}^{\alpha_n}(\omega^*, q_n)\big) = 2$, for all $n \ge n_1$.

In particular, with our choice of $\varepsilon_0 < ||A_0||_r$, we have

$$||A_n||_r \le ||A_n - A_0||_r + ||A_0||_r \le \frac{\varepsilon_0}{2} + ||A_o||_r \le 2||A_0||_r\,, \quad \text{for all } n \ge n_1\,. \tag{2.5}$$

Step (2) (Quantative perturbation argument I) : Next, apply Proposition (2.6) with $M = 2N$, (and without loss of generality we assume $\frac{M}{C_2} > 1$). This yields constants $\eta_0 \equiv \eta_0(2N)$ and $L \equiv L(N) > 0$ with the property described in that proposition.

Step (3) (Application of the open mapping theorem) : Fix any closed interval $J \subset (0,1)$ so that $\alpha \in J$. Now apply (II) of Proposition (2.7) with $M = 2||A_0||_r$, $\mathcal{F} = \{A_n \mid n \in \mathbb{N}\,, n > n_1\}$, $J' = \{\alpha_n \mid \alpha_n \in J\}$ and $\varepsilon = \frac{\varepsilon_0}{2}$ and with $M_1 = 2N$. Estimate (2.5) and a computation in Step (5) will show that the hypothesis of Proposition (2.7) is satisfied. So we get a constant $K \equiv K(||A_0||_r, N, r, \varepsilon_0, J)$ with the property stated in that proposition. Set,

$$L_* \equiv L_*(||A_0||_r, N, r, \varepsilon_0, J) = \frac{L}{K}\,.$$

Step (4) (Choosing a large q_ℓ) : Now we fix a value of n-say ℓ so large that $\ell > n_1$ and such that the following conditions hold.

$$\alpha_\ell \equiv \frac{p_\ell}{q_\ell} \in J\,, \tag{2.6}$$

$$\frac{K}{q_\ell^r} < \eta_0\,, \tag{2.7}$$

$$Cq_\ell^2||A_0||_r\exp\big(-(q_\ell^{r+1+\kappa} - 2||A_0||_r q_\ell)\big) < 1\,, \tag{2.8}$$

$$C{L_*}^2||A_0||_r q_\ell^{2r+2}\exp\big(-q_\ell^{r+1+\kappa} + 2L_* q_\ell^{r+1}||A_0||_r\big) < 1 \tag{2.9}$$

Step (5) (First application of Gronwall) : Verifying hypothesis of Proposition (2.6) and (2.7)) : Let $g = X^{\alpha_\ell}_{A_\ell}(\omega^*, q_\ell)$. Note that by (2.3), hypothesis of Lemma (2.8) is satisfied. Applying Lemma (2.8) with M replcaed by N and with $T = q_\ell$, we get the following estimate.

$$
\begin{aligned}
||g|| &\leq ||X^{\alpha_\ell}_{A_\ell}(\omega^*, q_\ell) - X^{\alpha}_{A_\ell}(\omega^*, q_\ell)|| + ||X^{\alpha}_{A_\ell}(\omega^*, q_\ell)|| \\
&\leq \frac{N(q_\ell)^2||A_\ell||_1}{2}\,|\alpha - \alpha_\ell|\exp(||A_\ell||_0 q_\ell) + N \\
&\leq N\frac{C(q_\ell)^2||A_\ell||_r}{2}\exp\big(-(q_\ell^{r+1+\kappa} - ||A_\ell||_r q_\ell)\big) + N|| \quad \text{by (1.7)} \\
&\leq NC(q_\ell)^2||A_0||_r\exp\big(-(q_\ell^{r+1+\kappa}|| - 2||A_0||_r q_\ell)\big) + N \qquad \text{by (2.5)}|| \\
&\leq N + N = 2N\,, \quad \text{by (2.8)}\,.
\end{aligned} \tag{2.10}
$$

Step (6) (Selecting a suitable perturbation) : Above estimate shows that $g = X^{\alpha_\ell}_{A_\ell}(\omega^*, q_\ell) \in P(2N)$. Hence we can apply Proposition (2.6), taking $\eta = \frac{K}{q_l^r}$, (note that $\eta < \eta_0$ by (2.7)). Thus we find a $g_1 \in SL(2,\mathbb{R})$ and $m \in \mathbb{N}$ such that

$$||g - g_1|| < \eta \quad \text{and} \quad ||g_1^m|| > 2N\,,$$

where m satisfies

$$m \leq \frac{L}{\eta} = \frac{Lq_\ell^r}{K} = L_* q_\ell^r\,, \tag{2.11}$$

recall that $L_* = \frac{L}{K}$.

Step (7) (Construction of B) : Next, apply Proposition (2.7), part (II), with choice of parameters as stated in Step (3). Take $g^* = g_1$. Note that the estimate in Step (5) shows that the hypothesis holds. Thus we get a map $B \in C^r(\Omega, sl(2,\mathbb{R}))$ such that
(a) $||B - A_\ell||_r < \frac{\varepsilon_0}{2}$ and
(b) $X^{\alpha_\ell}_B(\omega^*, q_\ell) = g_1$.
We shall show that the map B is the required map. We already have

$$||B - A_0||_r \leq ||B - A_\ell||_r + ||A_\ell - A_0||_r < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0\,.$$

Step (8) (Second application of Gronwall) : Suppose $B \notin E(N)$, this means

$$X^{\alpha}_B(\omega^*, t) \leq N\,, \quad \text{for all } t > 0\,. \tag{2.12}$$

We shall get a contradiction to this estimate. Since the rotation flow with rotation number α_ℓ is periodic with period q_ℓ, we have

$$X^{\alpha_\ell}_B(\omega^*, mq_\ell) = (X^{\alpha_\ell}_B(\omega^*, q_\ell))^m = g_1^m\,.$$

Thus,

$$||X^{\alpha_\ell}_B(\omega^*, mq_\ell)|| = ||g_1^m|| > 2N\,. \tag{2.13}$$

Now using Gronwall, we show that $||X^{\alpha}_B(\omega^*, mq_\ell)|| > N$. This contradicts (2.12) and shows that $B \in E(N)$, thus completing the proof.

First note that $\text{Max}\{||B||_0\,, ||B||_1\} \leq ||B||_r \leq ||A_0||_r + \varepsilon_0 \leq 2||A||_r$. Now applying Lemma (2.8) with $A = B$, $M = N$ and $T = mq_\ell$, estimate (2.2) yields

$$
\begin{aligned}
||X^{\alpha}_B(\omega^*, mq_\ell) - X^{\alpha_\ell}_B(\omega^*, mq_\ell)|| &\leq \frac{N(mq_\ell)^2||B||_1}{2}\,|\alpha - \alpha_\ell|\exp\big(mq_\ell||B||_0\big) \\
&\leq N(mq_\ell)^2||A_0||_r\big|\,\alpha - \alpha_\ell\,\big|\exp\big(2mq_\ell||A_0||_r\big)\,.
\end{aligned}
$$

Using estimate (2.11) we get

$$\begin{aligned}||X_B^\alpha(\omega^*, mq_n)-X_B^{\alpha_\ell}(\omega^*, mq_n)|| &\le NL_*^2||A_0||_r\, q_\ell^{2r+2}\big|\alpha-\alpha_\ell\big|\exp\big(2L_*q_\ell^{r+1}||A_0||_r)\big) \\ &\le NCL_*^2||A_0||_r q_\ell^{2r+2}\exp\big(-q_\ell^{r+1+\kappa}+2L_*q_\ell^{r+1}||A_0||_r\big)\,, \qquad \text{by (1.7)} \\ &\le N\,, \quad (\text{by } 2.9)\,.\end{aligned}$$

This estimate along with estimate (2.13) implies that $||X_B^\alpha(\omega^*, mq_\ell)|| > N$, which is what we wanted to prove to get a contradiction.

3 Proof of Theorem (1.9)

We begin by introducing some notation. We shall think of the real projective 1-space-$\mathbb{P}$-as the semicircle $\{e^{i\theta} \mid 0 \le \theta \le \pi\}$ whose 'ends' 1 and -1 are identified.

(a) Let d denote the 'usual angular metric' on $\mathbb{P}$, (i.e. $d(e^{i\theta_1}, e^{i\theta_2}) = \inf\{|\theta_1-\theta_2|, |\pi-(\theta_1+\theta_2)|\}$, $\theta_1, \theta_2 \in [0,\pi)$). Without loss of generality we shall assume that if $[v_1],[v_2] \in \mathbb{P}$, then

$$d([v_1]\,,[v_2]) \le D\,||\frac{v_1}{||v_1||}-\frac{v_2}{||v_2||}||\,,$$

where $D>0$ is some (universal) constant and $||\,||$ is the usual Euclidean norm on $\mathbb{R}^2$.

(b) Let

$$\mathcal{U}_r = \{A \in C^r(\Omega, sl(2,\mathbb{R})) \mid X_A^\alpha \text{ is uniformly hyperbolic}\}\,.$$

(c) For a given $N \in \mathbb{N}$ and $\delta > 0$ define the set

$$F(N,\delta) = \mathcal{U}_r \cup \tilde{F}(N,\delta)\,,$$

where

$\tilde{F}(N,\delta) = \{A \in C^r(\Omega, sl(2,\mathbb{R})) \mid \exists\, t_i \in \mathbb{R}\,, |t_i| > N\,, (i=1,2),$ and an open set $W_\delta(A) \subset \mathbb{P}$ with $\operatorname{diam}\big(W_\delta(A)\big) < \delta$ such that if $[v_1]\,,[v_2] \notin W_\delta(A)$ then $d\big([X_A^\alpha(\omega^*, t_j)v_1], [X_A^\alpha(\omega^*, t_j)v_2]\big) < \delta\,,$ for either $j=1$ or for $j=2\}$.

First, we claim that if $A \in \cap_{n\in N}F(n,\frac{1}{n})$ then either X_A^α is uniformly hyperbolic or X_A^α is proximal. Suppose X_A^α is not uniformly hyperbolic. Let $[v_1],[v_2] \in \mathbb{P}$, $[v_1] \ne [v_2]$. Let $n_1 \in \mathbb{N}$ be such that $d([v_1],[v_2]) > \frac{1}{n}$ for all $n \ge n_1$. Thus for each $n \ge n_1$, since $A \in F(n,\frac{1}{n})$, there is some t_n, $t_n > n$ such that $d([X_A^\alpha(\omega^*, t_n)v_1]\,,[X_A^\alpha(\omega^*, t_n)v_2]) \le \frac{1}{n}$. This shows that all distinct points in the fiber over ω^* are proximal. By the minimality of the base flow, it follows that the skew product projective flow generated by the cocycle X_A^α is a proximal extension, i.e. X_A^α is proximal, (see [4]).

Thus, our theorem now follows from the Baire category argument once we show that each $F(N,\delta)$ is open and dense in $C^r(\Omega, sl(2,\mathbb{R}))$. Openness of the set $\mathcal{U}_r$ in $C^r(\Omega, sl(2,\mathbb{R}))$ is a general fact about uniform hyperbolicity and the openness of the set $\tilde{F}(N,\delta)$ is a consequence of continuity of the map $A \to X_A^\alpha(\omega^*, t)$, (for a fixed t) and the compactness of $\mathbb{P}\times\mathbb{P}\backslash W_\delta(A)\times W_\delta(A)$. Now we prove that each $F(N,\delta)$ is dense.

So, let $N \in \mathbb{N}$, $\delta > 0$, $A_0 \in C^r(\Omega, sl(2,\mathbb{R}))$ and $\varepsilon_0 > 0$ be given. We shall suppose that X_A^α is not uniformly hyperbolic for any A that is within ε_0 neighbourhood of A_0. We shall construct a function $B \in C^r(\Omega, sl(2,\mathbb{R}))$ such that

(i) $||B - A_0||_r < \varepsilon_0$, and
(ii) $B \in \tilde{F}(N, \delta)$.

As before, the construction B will involve the same four 'ingradients'. However, now we need a different quantitative perturbation lemma to ensure proximality and we need a 'uniformity result' about perturbations of parabolic matrices. We begin by introducing a bit more notation.

(d) Recall that P denotes the set of parabolic matrices and one can parametrize P as $P = \{g = g_{\mu,\theta} \mid \mu \in \mathbb{R}, 0 \le \theta < \pi\}$. Note that the unique eigenvector of $g_{\mu,\theta}$ with standard Euclidian norm 1, is the vector $\begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix}$.

(e) Given a $\delta > 0$, let $W_\delta(g) \equiv W_\delta(g_{\mu,\theta})$ denote the open δ ball in $\mathbb{P}$ centered at the ray determined by the eigenvector of $g_{\mu,\theta}$.

Proposition 3.1 *(A quantitative perturbation lemma II) Let $N \in \mathbb{N}$ and $\delta \in (0, \frac{\pi}{2})$ be given. Then there exists a constant $D^* \equiv D^*(\delta) > 0$ such that given any 'small enough' $\eta > 0$, (more precisely $\eta \in (0, \eta^*)$ where $\eta^* = \frac{4}{N\, tan(\delta)}$), there exists a positive integer $m \in \mathbb{N}$ such that*

(1) $N < m$,

(2) $m \le \frac{D^*}{\eta}$, and

(3) the following holds : given any $g \in P$, we can find a $g_1 \in P$ such that

(3_a) $||g - g_1|| < \eta$, and

(3_b) if $[v_1], [v_2] \notin W_\delta(g)$, then $d\big([g_1^k v_1], [g_1^k v_2]\big) < \delta$, for either $k = m$ or $k = -m$.

Proof: We want to find a constant $D^* \equiv D^*(\delta)$ and $m \equiv m(\delta, \eta) \in \mathbb{N}$ such that the three properties listed in the lemma hold. We begin by considering $g_\mu \equiv g_{\mu,0} \in P$ and shall obtain D^* and m so that properties (1) and (2) will hold and property (3) will hold for g's of this form. Then we show that the same D^* and m 'will work' for any general $g \equiv g_{\mu,\theta}$ in P.

So consider a $g_\mu \equiv g_{\mu,0} \in SL(2,\mathbb{R})$ and a point $e^{i\theta} \in \mathbb{P}$, $(0 \le \theta < \pi)$. Let $\theta_{g_\mu} \in [0, \pi)$ be the angle determined by the ray corresponding to the vector $g_\mu \begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix}$. Note that

$$tan(\theta_{g_\mu}) = \frac{1}{\frac{1}{tan(\theta)} + \mu},$$

(we treat ∞ and 0 as reciprocals of each other). We introduce the following subsets of $\mathbb{P}$,

$$S_\delta^+ = \{e^{i\varphi} \mid 0 \le \varphi < \delta\} \quad \text{and} \quad S_\delta^- = \{e^{i\varphi} \mid \pi - \delta < \varphi \le \pi\}.$$

Note that, $W_\delta(g_{\mu,0}) = S_\delta^+ \cup S_\delta^-$.

Suppose $\mu \ge 0$ and $e^{i\theta} \notin S_\delta^-$. Then, $-\frac{1}{tan(\delta)} < \frac{1}{tan(\theta)}$, Given $\eta \in (0, \eta^*)$, select $\mu_1 > 0$ so close to μ that $||g_\mu - g_{\mu_1}|| < \eta$ and $\frac{\eta}{2} < \mu_1$. Our g_1 will be g_{μ_1}, thus (3_a) holds. Next, for any $n \in \mathbb{N}$,

$$tan(\theta_{g_{\mu_1}^n}) = tan(\theta_{g_{n\mu_1}}) = \frac{1}{\frac{1}{tan(\theta)} + n\mu_1} \le \frac{1}{\frac{1}{-tan(\delta)} + n\mu_1} < \frac{1}{\frac{1}{-tan(\delta)} + n\frac{\eta}{2}}.$$

Thus if we choose $m \in \mathbb{N}$ to be the first positive integer n so that $-\frac{1}{tan(\delta)} + n\frac{\eta}{2} > \frac{1}{tan(\delta)}$, i.e. $n > \frac{4}{\eta tan(\delta)}$, then

(i) $m > \frac{4}{\eta \tan(\delta)} > \frac{4}{\eta^* tan(\delta)} = N$ and
(ii) $m \leq \frac{4}{\eta \tan(\delta)} + 1 \leq \frac{4}{\eta tan(\delta)} + \frac{4}{\eta tan(\delta)} = \frac{8}{\eta tan(\delta)}$, (note that $1 \leq N < \frac{4}{\eta tan(\delta)}$). Thus (2) holds with $D^* = \frac{8}{tan(\delta)}$. Also note that D^* and m are independant of $g \in P$ as long as g has the form $g = g_{\mu,0}$.
(iii) Now we verify property (3_b). Since $W_\delta(g_{\mu,0}) = S_\delta^+ \cup S_\delta^-$, by our choice of m, $tan(\theta_{g_{\mu_1}^m}) < tan(\delta)$, which means that $g_{\mu_1}^m$ maps the complement of S_δ^- into S_δ^+. In particular, if $[v_1], [v_2] \notin S_\delta^-$ then $d([g_{\mu_1}^m v_1], [g_{\mu_1}^m v_2]) < \delta$. Similarly, if $[v_1], [v_2] \notin S_\delta^+$ then $d([g_{\mu_1}^{-m} v_1], [g_{\mu_1}^{-m} v_2]) < \delta$. Thus, we have proved that (3_b) holds. If $\mu < 0$, a similar argument can be given interchanging the roles of m and $-m$.

Now we consider the case of a general $g \in P$. We show that the chosen D^* and m will also 'work' for this g as well, i.e. (3) of this lemma remains valid for this g. As seen before, such a g can be written as

$$g \equiv g_{\mu,\varphi} = R(\varphi) g_{\mu,0} R(\varphi)^{-1},$$

where $R(\varphi)$ is the rotation matrix corresponding to rotation by angle φ. Since $R(\varphi)$ is angle and norm preserving, our previous analysis remains valid for $g_{\mu,\varphi}$ as well. Again we shall assume that $\mu \geq 0$, (other case is similar). Select μ_1 close enough to μ so that $||g_{\mu,\varphi} - g_{\mu_1,\varphi}|| < \eta$ and $\mu_1 > \frac{\eta}{2}$, (this is possible since $R(\varphi)$ is norm preserving). Take $W_\delta(g_{\mu,\varphi}) = R(\varphi)(W_\delta(g_{\mu,0}))$. Since $R(\varphi)$ is angle preserving, it follows that g^m or g^{-m} will map all points not in $W_\delta(g_{\mu,\varphi})$ within δ of each other. Thus (3) holds. ■

Next, we also need a uniformity lemma. Note that, for a fixed $v \in \mathbb{R}^2$, the map $g \to gv : SL(2,\mathbb{R}) \to \mathbb{R}^2$ is uniformly continuous because of linearity of the action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$, but this is not so for the map $g \to [gv] : SL(2,\mathbb{R}) \to \mathbb{P}$. For the approximation argument we shall need the following.

Lemma 3.2 *Let $g \in P$, $[v] \notin W_\delta(g)$ and $h \in SL(2,\mathbb{R})$, then*

$$d([gv],[hv]) \leq \frac{2D}{|sin(\delta)|} ||g-h||.$$

Proof: Let $g \in P$, $[v] \notin W_\delta(g)$, $||v|| = 1$ and $h \in SL(2,\mathbb{R})$, consider

$$\begin{aligned} d([gv],[hv]) &\leq D|| \frac{gv}{||gv||} - \frac{hv}{||hv||} || \leq \frac{D}{||gv||\,||hv||} \big|\big| \, ||hv||\, gv - ||gv||\, hv \big|\big| \\ &\leq \frac{D}{||gv||\,||hv||} \left[||hv||\;||gv-hv|| + \big|\, ||hv|| - ||gv|| \,\big|\, ||hv|| \right] \\ &\leq \frac{D}{||gv||} \left[||g-h||\,||v|| + ||g-h||\,||v|| \right] = \frac{2D}{||gv||} ||g-h||. \end{aligned} \tag{3.1}$$

Now let $g \equiv g_{\mu,\theta}$. Since $v \notin W_\delta(g) = R(\theta) W_\delta(g_{\mu,0})$, hence $R(\theta)^{-1} v \notin W_\delta(g_{\mu,0})$. Thus letting $[v] = R(\theta) \begin{pmatrix} cos(\varphi) \\ sin(\varphi) \end{pmatrix} = \begin{pmatrix} cos(\varphi+\theta) \\ sin(\varphi+\theta) \end{pmatrix}$, it follows that $|sin(\varphi)| > |sin(\delta)|$. Thus,

$$||gv||^2 = ||R(\theta) g_{\mu,0} R(\theta)^{-1} v||^2 = || \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\varphi) \\ sin(\varphi) \end{pmatrix} ||^2 \geq |sin(\varphi)|^2 \geq |sin(\delta)|^2.$$

Thus, $d([gv],[hv]) \leq \frac{2D}{|sin(\delta)|} ||g-h||$. ■

Proof of the density of $F(N,\delta)$

Proof: Thus, $N \in \mathbb{N}$ and $\delta > 0$ are given, (without loss of generality, we shall assume that $\delta \in (0, \frac{\pi}{2})$). To prove the density, we shall let $A_0 \in C^r(\Omega, sl(2,\mathbb{R}))$, $\varepsilon_0 > 0$ and suppose that X_A^α is

not uniformly hyperbolic for any A that is within the ε_0 neighbourhood of A_0. We shall construct a function $B \in C^r(\Omega, sl(2,\mathbb{R}))$ such that
(i) $||B - A_0||_r < \varepsilon_0$, and
(ii) $B \in \tilde{F}(N, \delta)$.

Step (1) (The Rotation number argument) : Since X_A^α is not uniformly hyperbolic for any A that is within the ε_0 neighbourhood of A_0, applying Proposition (2.5) to A_0 with $\gamma = \alpha$, $\gamma_n = \alpha_n \equiv \frac{p_n}{q_n}$, $\varepsilon = \frac{\varepsilon_0}{2}$ and $\xi = \frac{\varepsilon_0}{2}$, we get an $n_1 \in \mathbb{N}$ and an infinite sequence $A_n \in C^r(\Omega, sl(2,\mathbb{R}))$, $n \geq n_1$, such that
(i) $||A_0 - A_n|| < \frac{\varepsilon_0}{2}$, for all $n \geq n_1$ and
(ii) $tr\big(X_{A_n}^{\alpha_n}(\omega^*, q_n)\big) = 2$, for all $n \geq n_1$.

As in the proof of density of $E(N)$, without loss of generality, we shall assume that $\varepsilon_0 < ||A_0||_r$ and we get the estimate $||A_n||_r \leq 2||A_0||_r$ for all $n \geq n_1$.

Step (2) (Qualitative perturbation lemma II) : Next, apply Proposition (3.1) with the given N and δ. Let $\eta_0 = \frac{4}{N\, tan(\delta)}$. Then Proposition (3.1) yields for every $\eta \in (0, \eta_0)$, a constant $D^* \equiv D^*(\delta)$ and a positive integer $m \equiv m(\delta, \eta) \in \mathbb{N}$ with the properties described in Proposition (3.1). The choice of η will be made in Step (5).

Step (3) (Application of the open mapping theorem) : Fix any closed interval $J \subset (0,1)$ so that $\alpha \in J$. Now apply (II) of Proposition (2.7) with $M = 2||A_0||_r$, $\mathcal{F} = \{A_n \mid n \in \mathbb{N}, n > n_1\}$, $J' = \{\alpha_n \mid \alpha_n \in J\}$ and $\varepsilon = \frac{\varepsilon_0}{2}$ and with $M_1 = C_1 \frac{\eta_0}{2}$. We shall see later that the hypothesis of Proposition (2.7) is satisfied, (see the comment in Step (5)). So we get a constant $K \equiv K(||A_0||_r, N, \delta, r, \varepsilon_0, J)$ with the property stated in that proposition.

Step (4) (Choosing a large q_ℓ) : Now we fix a value of n-say ℓ so large that $\ell > n_1$ and such that the following conditions hold.

$$\alpha_\ell \equiv \frac{p_\ell}{q_\ell} \in J\,, \tag{3.2}$$

$$\frac{K}{q_\ell^r} < \eta_0\,, \tag{3.3}$$

$$\frac{C||A_0||_r D^{*2} q_\ell^{2r+2}}{K^2} \exp\Big(\frac{4||A_0||_r D^* q_\ell^{r+1}}{K} - q_\ell^{r+1+\kappa}\Big) < \frac{\delta}{3} \cdot \frac{|sin(\delta)|}{2D}\,. \tag{3.4}$$

Step (5) (Selecting a suitable pertubation) : Apply Proposition (3.1) with $\eta = \frac{K}{q_\ell^r} < \eta_0$ and δ replaced by $\frac{\delta}{3}$. Thus we get an integer $m \equiv m(\delta, \eta)$ so that the conclusions of that Proposition hold. Thus, (i) $N < m$, (ii) $m \leq \frac{D^*}{K} q_\ell^r$ and (iii) taking $g = X_{A_\ell}^{\alpha_\ell}(\omega^*, q_\ell) \in P$, we find a $g_1 \in P$ that satisfies (3_a) and (3_b) of Proposition (3.1).

At this point we remark that if we look at the proof of Proposition (3.1), we need to change g to g_1 only if $|\mu| \leq \frac{\eta}{2}$ where $g = X_{A_\ell}^{\alpha_\ell}(\omega^*, q_\ell) = g_{\mu,\theta}$. That is, properties (1) to (3) of Proposition (3.1) are valid for $g \equiv g_{\mu,\theta}$ itself if $|\mu| \geq \frac{\eta_0}{2}$. This means that the case when we need to apply Proposition (3.1) to get a perturbation g_1 is when $g = X_{A_\ell}^{\alpha_\ell}(\omega^*, q_\ell) = g_{\mu,\theta}$ has norm less than or equal to $Max\{C_1\,, C_1\frac{\eta_0}{2}\}$. Thus, hypothesis of Proposition (2.7) part II holds with $M_1 = Max\{C_1\,, C_1\frac{\eta_0}{2}\}$.

Step (6) (Construction of B) : Next, apply Proposition (2.7) with the same choice of parameters as in Step (3). Take $g^* = g_1$, where $g_1 \in P$ is the element obtained in Step (5). Thus, we get $B \in C^r(\Omega, sl(2,\mathbb{R}))$ such that
(a) $||B - A_\ell||_r < \frac{\varepsilon_0}{2}$ and
(b) $X_B^{\alpha_\ell}(\omega^*, q_\ell) = g_1$.

We shall show that the map B is the required map.

Step (7) (Verifying that $B \in F(N,\delta)$) : Recall that $g = X_{A_\ell}^{\alpha_\ell}(\omega^*, q_\ell) \in P$ and $g_1 = X_B^{\alpha_\ell}(\omega^*, q_\ell) \in P$ and note that g_1 obtained in Step (5) by applying Proposition (3.1) have the same eigen vector. Thus $W_\delta(g_1) = W_\delta(g)$, now take $W_\delta(B)$ to be $W_\delta(g_1)$. We shall verify that if $[v_1],[v_2] \notin W_\delta(B)$ then $d([X_B^\alpha(\omega^*,t)v_1],[X_B^\alpha(\omega^*,t)v_2]) < \delta$ for either $t = mq_\ell$ or $t = -mq_\ell$. This will prove that $B \in \tilde{F}(N,\delta)$. Since g_1 satisfies (3_b) of Proposition (3.1), we know that if $[v_1],[v_2] \notin W_\delta(B)$, then

$$d([X_B^{\alpha_\ell}(\omega^*, q_\ell)^k v_1],[X_B^{\alpha_\ell}(\omega^*, q_\ell)^k v_2]) < \frac{\delta}{3} \quad \text{for either } k = m \text{ or } k = -m\,.$$

Note that $X_B^{\alpha_\ell}(\omega^*, \pm mq_\ell) = X_B^{\alpha_\ell}(\omega^*, q_\ell)^{\pm m}$, since the flow with winding number α_ℓ is periodic with period q_ℓ. Now suppose

$$||X_B^\alpha(\omega^*, mq_\ell) - X_B^{\alpha_\ell}(\omega^*, mq_\ell)|| \leq \frac{\delta}{3}\,\frac{|sin(\delta)|}{2D}\,. \tag{3.5}$$

Using $W_\delta(B) = W_\delta(g_1) = W_\delta(g)$, lemma (3.2) shows that if $[v_1],[v_2] \notin W_\delta(B)$, then for either $k = m$ or $k = -m$,

$$d([X_B^\alpha(\omega^*, \pm mq_\ell)v_i],[X_B^{\alpha_\ell}(\omega^*, \pm mq_\ell)v_i]) < \frac{\delta}{3} \quad \text{for } i = 1,2\,.$$

Thus, by the triangle inequality, (for metric d on $\mathbb{P}$), it follows that $B \in \tilde{F}(N,\delta)$. So the proof boils down to checking estimate (3.5), which will result from the Gronwall's inequality along with the choice of q_ℓ made in (3.4). We cannot use the previous version of Gronwall directly because now our cocycle $X_B^\alpha(\omega^*,t)$ is not bounded for all $t > 0$. However this is not a problem because we need only an a priori bound on $X_B^\alpha(\omega^*,t)$ for $t \in [0, mq_\ell]$. First note that $||B||_i \leq ||B - A_0||_i + ||A_0||_i \leq \varepsilon_0 + ||A_0||_i \leq 2||A_0||_r$, for $1 \leq i \leq r$, (without loss of generality we have assumed that $\varepsilon_0 < ||A_0||_r$). Let $\gamma \in [0,1]$ be any winding number. Note that $x(t) = X_B^\gamma(\omega^*,t)x_0$ satisfies the integral equation $x(t) = x_0 + \int_0^t B(T_s^\gamma\omega^*)x(s)ds$. Hence the estimate $||x(t)|| \leq ||x_0|| + \int_0^t ||B(T_s^\gamma\omega^*)||\,||x(s)||ds$ allows one to apply Gronwall's inequality which yields the estimate $||X_B^\gamma(\omega^*,t)x_0|| \leq ||x_0||e^{||B||_0 t} \leq ||x_0||e^{2||A_0||_r t}$ for any $t > 0$. Taking the supremum over all $||x_0||$'s of norm one yields

$$||X_B^\gamma(\omega^*,t)|| \leq e^{2||A_0||_r t}\,. \tag{3.6}$$

for all $t > 0$. Thus, now applying inequality (2.2) with $A = B$ and where M is replaced by the bound $e^{2||A_0||_r T}$, where $T = mq_\ell$, we get

$$\begin{aligned}
||X_B^\alpha(\omega^*, mq_\ell) - X_B^{\alpha_\ell}(\omega^*, mq_\ell)|| &\leq \frac{||B||_1 m^2 q_\ell^2}{2} e^{2||A_0||_r m q_\ell}\big|\alpha - \alpha_\ell\big| e^{||B||_0 m q_\ell} \\
&\leq \frac{||A_0||_r (D^*)^2 q_\ell^{2r+2}}{K^2}\exp\Big(\frac{4||A_0||_0 D^* q_\ell^{r+1}}{K}\Big)\big|\alpha - \alpha_\ell\big| \quad (\text{since } m \leq \frac{D^* q_\ell^r}{K})\,, \\
&\leq \frac{C||A_0||_r (D^*)^2 q_\ell^{2r+2}}{K^2}\exp\Big(\frac{4||A_0||_r D^* q_\ell^{r+1}}{K} - q_\ell^{r+1+\kappa}\Big) \\
&\leq \frac{\delta}{3}\,\frac{|sin(\delta)|}{2D}\,, \quad \text{by (3.4)}\,.
\end{aligned}$$

This establishes the desired inequality (3.5) and the proof is complete. ■

4 Proof of Proposition (2.7)

We begin with some notation and observations.

(1) We shall think of Ω as $\mathbb{R}^2/\mathbb{Z}^2$. Let $e : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be the exponential mapping $e(x,y) = (e^{2\pi ix}, e^{2\pi iy})$. Given a rational rotation number $\alpha_n = \frac{p_n}{q_n} \in J$, consider flows $\{\tilde{T}_t^{\alpha_n}\}_{t\in\mathbb{R}}$ and $\{\tilde{T}_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}^2$ defined by

$$\tilde{T}_t^{\alpha_n}(x,y) = (x + \alpha_n t, y + t)\,,$$
$$\tilde{T}_t(x,y) = (x, y + t)\,.$$

The flow $\{\tilde{T}_t^{\alpha_n}\}_{t\in\mathbb{R}}$ projects on to the flow $\{T_t^{\alpha_n}\}_{t\in\mathbb{R}}$ on $\mathbb{T}^2$ and has a 'natural fundamental domain' $F_n \subset \mathbb{R}^2$ given by the parallalogram

$$F_n = \{\tilde{T}_t^{\alpha_n}(x,0) \mid -\frac{1}{2q_n} \le x \le \frac{1}{2q_n}\,, \quad 0 \le t \le 1\}\,.$$

Note that F_n is also a flow box and it is a 'fundamental domain' in the sense that the first q_n iterates of $e(F_n) \subset \mathbb{T}^2$, under the time one map of the flow $(\Omega, \{T_t^{\alpha_n}\}_{t\in\mathbb{R}})$ 'partitions' Ω. Note that $e : F_n \to e(F_n)$ is a C^r diffeomorphism.

(2) Now, given a $\alpha_n \in J$ and a map $A : \Omega \to sl(2,\mathbb{R})$, our perturbation of A will be of the form $A + B$ where $B; \Omega \to sl(2,\mathbb{R})$ and support of B will be in the interior of $e(F_n)$. It is convinient to construct the perturbation on F_n and then define it on $e(F_n)$ via composition with e^{-1}. In fact, for clarity it is even better to 'straighten out' the parallalogram F_n and replace it by the rectangle $R_n \subset \mathbb{R}^2$ defined by

$$R_n = [-\frac{1}{2q_n}, \frac{1}{2q_n}] \times [0,1]\,.$$

Clearly R_n is a flow box for the flow $\{\tilde{T}_t\}_{t\in\mathbb{R}}$ defined above and the map $H_n : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$H_n(x,y) = (x + \alpha_n y, y)\,,$$

is a diffeomorphism, maps R_n diffeomorphically onto F_n and also intertwines flows $\{\tilde{T}_t\}_{t\in\mathbb{R}}$ and $\{\tilde{T}_t^{\alpha_n}\}_{t\in\mathbb{R}}$, i.e.

$$H_n \circ \tilde{T}_t(x,y) = \tilde{T}_t^{\alpha_n} \circ H_n(x,y)\,, \quad \text{for } x, y \in R_n\,, \text{ and } 0 \le t \le 1\,.$$

(3) Thus, given a $\alpha_n \in J$ and a map $A : \Omega \to sl(2,\mathbb{R})$, let $a : R_n \to sl(2,\mathbb{R})$ be the map

$$a = A \circ E_n\,, \quad \text{where} \tag{4.1}$$
$$E_n = e \circ H_n : R_n \to \mathbb{T}^2\,, \tag{4.2}$$

(i.e. the small case letters will denote composition with E_n of a map denoted by the corresponding capital letter). Note that E_n is a diffeomorphism from R_n onto its image $e(F_n)$, maps $(0,0)$ to ω^* and it intertwines flows $\{\tilde{T}_t\}_{t\in\mathbb{R}}$ and $\{T_t^{\alpha_n}\}_{t\in\mathbb{R}}$. In particular, considering just the orbit of ω^*, this implies that

$$X_A^{\alpha_n}(\omega^*, t) = X_a(t)\,, \quad \text{for all } t \in [0,1]\,, \tag{4.3}$$

where $X_a(t) : [0,1] \to SL(2,\mathbb{R})$ is the solution to the initial value problem $X' = a(\tilde{T}_t(0,0))X = a(0,t)X$, $X(0) = I$.

(4) Recall that given $M > 0$, $\varepsilon > 0$, $J \subset (0,1)$ and a family $\mathcal{F} \subset \mathcal{F}_M \equiv \{A \in C^r(\Omega, sl(2,\mathbb{R})) \mid ||A||_r \leq M\}$, for any $A_0 \in \mathcal{F}$ and $\gamma_n = \frac{p_n}{q_n} \in J$ "if g^* 'sufficiently close" to $X^{\gamma_n}_{A_0}(\omega^*, q_n)$, then we want to find a perturbation (i.e. a map B_0) within ε of A_0 such that $X^{\gamma_n}_{B_0}(\omega^*, q_n) = g^*$. We shall consider $a_0 = A_0 \circ E_n$ and find its perturbation $b_0 : R_n \to sl(2,\mathbb{R})$ supported on R_n. Then $B_0 = b_0 \circ E_n^{-1}$ will be the required perturbation of A_0. Note that $(A_0 + B_0) \circ E_n = A_0 \circ E_n + B_0 \circ E_N = a_0 + b_0$ and hence by (4.3) we have

$$X^{\gamma_n}_{A_0+B_0}(\omega^*, 1) = X_{a_0+b_0}(1)\,.$$

(5) The perturbed map b_0 will be of the form

$$b_0(x,y) = a_0(x,y) + b_1(x)b_2(y)\,,$$

where $b_1 : [-\frac{1}{2q_n}, \frac{1}{2q_n}] \to \mathbb{R}$ and $b_2 : [0,1] \to sl(2,\mathbb{R})$. Again, recall that $[0,1]$ is identified with $\{(0,y) \in R_n \mid y \in [0,1]\} = \{\hat{T}_t(0,0) \mid 0 \leq t \leq 1\}$. We shall condsider the construction of b_2 on this set shortly, whereas map b_1 is just a smooth bump function which satisfies
(i) b_1 is C^r,
(ii) $b_1(0) = 1$ and
(iii) support of b_1 is compact and contained in $(-\frac{1}{4q_n}, \frac{1}{4q_n})$.

To construct b_1, take any C^r bump function, say ψ, with compact support inside $(-1,1)$ and dilate it, i.e. set $b_1(t) = \psi(4q_n t)$, for $t \in [-\frac{1}{2q_n}, \frac{1}{2q_n}]$. Thus $||b_1||_r \leq K_4(r)q_n^r$ where K_4 is some universal constant that depends on ψ and hence only on r.

(6) Before describing the construction of b_2, let us consider some estimates. First, the product rule implies

$$||A_0 - B_0||_r = ||(a_0 - b_0) \circ E_n^{-1}||_r \leq K_1(r)||E_n^{-1}||_r||a_0 - b_0||_r\,,$$

where K_1 is some constant that depends only on r. Furthermore $||E_n^{-1}||_r \leq K_2(r,J)$, where K_2 is a constant that depends on r and J alone, (as long as J is compact and does not contain 0 or 1, first r derivarives of E_n^{-1} are bounded by a constant that depend only on J and r). Thus,

$$||A_0 - B_0||_r = K_1K_2||a_0 - b_0||_r\,.$$

Now,

$$||a_0 - b_0||_r = ||b_1b_2||_r \leq K_3||b_1||_r\;||b_2||_r\,,$$

where $K_3 \equiv K_3(r)$ is a constant dependent only on r, (again due to the product rule). We already saw that $||b_1||_r \leq K_4q_n^r$. Thus we have

$$||A_0 - B_0||_r \leq K_1K_2K_3K_4q_n^r||b_2||_r,\,,$$

where the constants K_i, $(1 \leq i \leq 4)$ depend only on r and J.

(7) Finally, we have to construct map $b_2 : [0,1] \to sl(2,\mathbb{R})$ and this map will depend on the choice of point g^* sufficiently close to $X^{\gamma_n}_{A_0}(\omega^*, q_n)$. Existence of such a map is a consequence of an open mapping theorem which we shall formulate now.

Consider the space $C^r([0,1], sl(2,\mathbb{R}))$-the space of all C^r curves From $[0,1]$ to $sl(2,\mathbb{R})$ with the C^r metric. For $f \in C^r([0,1], sl(2,\mathbb{R}))$, and $k,l \in \mathbb{N}\cup\{0\}$ with $k < l$, by a '(k,l)-jet of f at x' we shall mean the $l-k+1$ vector $(f^{(k)}(x), \cdots, f^{(l)}(x))$. Our perturbation $b_1(x)b_2(y)$ is supported on R_n, b_1 is a compactly supported C^r bump function, so to make the map $b_0 = a_0 + b_1b_2$ a C^r map, we need to make sure that b_2 is C^r and the $(0,r)$-jet of b_2 at 0 and 1 is zero (i.e. the zero matrix). So given any $a \in \mathcal{F}_M \equiv \{f \in C^r([0,1], sl(2,\mathbb{R})) \mid ||f||_r \le M\}$ and $\delta > 0$, define

$$N^r(a,\delta) = \{b \in C^r(\Omega, sl(2,\mathbb{R})) \mid ||a-b||_r < \delta \text{ and } a \text{ and } b \text{ have the same } (0,r)- \text{ jet at 0 and 1}\}.$$

For each $b \in C^r([0,1], sl(2,\mathbb{R}))$, let $t \to X_b(t) : [0,1] \to SL(2,\mathbb{R})$ be the solution to the initial value problem $X' = b(t)X$, $X(0) = I$, where I is the identity matrix. Define $\Psi : C^r([0,1], sl(2,\mathbb{R})) \to SL(2,\mathbb{R})$ by setting

$$\Psi(b) = X_b(1).$$

Then we have the following.

Lemma 4.1 *Let $M > 0$. There exists a constant $K^* \equiv K^*(M,r)$ with the following property : given any $\delta > 0$, for any $a \in \mathcal{F}_M$, the image of the set $N^r(a,\delta)$ under the map Ψ contains the ball*

$$B_{d^*}(\Psi(a), K^*\delta) = \{g \in SL(2,\mathbb{R}) \mid d^*(g, \Psi(a)) < K^*\delta\}.$$

(8) Using this lemma, first we shall finish the proof of Theorem (2.7). Recall that $r \in \mathbb{N}$, $M > 0$, $J \subset (0,1)$ and $\varepsilon > 0$, along with a family $\mathcal{F} \subset \mathcal{F}_M$ is given. Let $A_0 \in \mathcal{F}$ and $\gamma_n \in J$. As discussed before, let $a_0 = A_0 \circ E_n$, our perturbed map B_0 will be of the form $B_0 = b_0 \circ E_n^{-1}$ where $b_0(x,y) = a_0(x,y) + b_1(x)b_2(y)$. To select map b_2 we apply lemma (4.1) with $a = a_0$ and $\delta = \frac{\varepsilon}{K_1K_2K_3K_4q_n^r}$. We shall see that the constant K in the conclusion of Proposition (2.7) will turn out to be $K \equiv K(M,r,\varepsilon,J) = \frac{K^*}{K_1K_2K_3K_4}\varepsilon$, (note that K_i's depend on r and J, K^* depends on M and r and in fact the dependence of K on ε is linear).

To verify the conclusion (a) and (b) of Proposition (2.7), let $g^* \in SL(2.R)$ be such that

$$d^*(g^*, X_{A_0}^{\gamma_n}(\omega^*, q_n)) < \frac{K}{q_n^r}.$$

Let $h = X_{A_0}^{\gamma_n}(T_1\omega^*, q_n - 1))^{-1}g^*$, then

$$\begin{aligned} d^*(h\,, \Psi(a_0)) &= d^*(h, X_{a_0}(1)) = d^*(h, X_{A_0}^{\gamma_n}(\omega^*, 1)) \\ &= d^*(g^*, X_{A_0}^{\gamma_n}(T_1\omega^*, q_n - 1)X_{A_0}^{\gamma_n}(\omega^*, 1)), \quad \text{(by the left invariance of } d^*), \\ &= d^*(g^*, X_{A_0}^{\gamma_n}(\omega^*, q_n)) < \frac{K}{q_n^r}, \quad \text{(note that } \frac{K}{q_n^r} = K^*\delta). \end{aligned}$$

Select $a' \in N^r(a_0, \delta)$ so that $X_{a'}(1) = h$ and let $b_2 = a' - a$. Then, since $A_0(T_{1+t}\omega^*) = B_0(T_{1+t}\omega^*)$ for all $t \in [0, q_n - 1]$, it follows that $X_{B_0}(\omega^*, q_n) = g^*$. Finally,

$$||A_0 - B_0||_r \le K_1K_2K_3K_4q_n^r||b_2||_r \le K_1K_2K_3K_4q_n^r \times \frac{\varepsilon}{K_1K_2K_3K_4q_n^r} = \varepsilon,$$

by our choice of δ.
(II) This follows at once because if g_1 and g_2 are in the compact set $\{g \in SL(2,\mathbb{R}) \mid ||g|| \le M_1\}$, then $d^*(g_1, g_2) \le K_5||g_1 - g_2||$ for some constant K_5 that depends on M_1 alone. This completes the proof of Proposition (2.7).

Now we turn to the proof of Lemma (4.1).

Proof of Lemma (4.1)

Proof: Note that each $a \in \mathcal{F}_M$ is uniquely determined by X_a, namely $a = \frac{dX_a}{dt} X_a^{-1}$. Thus we shall construct a perturbation b of a given a by considering perturbation of X_a (i.e. of the curve $t \to X_a(t) : [0,1] \to SL(2,\mathbb{R})$), of the form $t \to X_a(t)g(t)$, where $t \to g(t) : [0,1] \to SL(2,\mathbb{R})$ is a suitable curve; and then take b to be $b = \frac{dX_a \cdot g}{dt}(X_a \cdot g)^{-1}$. Thus,

$$b(t) - a(t) = \frac{dX_a \cdot g}{dt}(X_a \cdot g)^{-1} - a(t) = X_a \frac{dg}{dt}(g)^{-1} X_a^{-1} \, . \tag{4.4}$$

This identity and the above discussion points out that there is a constant $K_1^* \equiv K_1^*(M,r)$ such that

$$||b - a||_r \leq K_1^* ||g||_{r+1} \, .$$

Now we need to show that there is a constant $K_2^* \equiv K_2^*(r)$ such that every point ξ that is within $K_2^* \delta$ of the identity in $SL(2,\mathbb{R})$ can be joined to the identity I by a smooth curve $g : [0,1] \to SL(2,\mathbb{R})$ such that

(i) $g^{(k)}(0) = g^{(k)}(1)$- the k^{th} derivative of g at 0 and 1 are zero i.e. (-the zero matrix), for $k = 1, 2, \cdots, r+1$ and

(ii) $||g||_{r+1} \leq \frac{\delta}{K_1^*}$.

Once this is proved, the proof of Lemma (4.1) follows by taking $K^* = \frac{K_2^*}{K_1^*}$. ■

Thus, we have reduced our problem to showing that : given a $\delta > 0$, there is a ball centered at I in $SL(2,\mathbb{R})$, of radius $K_2^* \delta$, (where $K_2^* \equiv K^*(r)$),such that every point in it can be joined to the identity matrix I by a C^r curve $t \to g(t)$ whose C^r norm is less than δ and whose first $r+1$ derivatives at 0 and 1 are 0. Now, these requirements on the curve $t \to g(t)$ are 'local' and $SL(2,\mathbb{R})$ is locally diffeomorphic to $\mathbb{R}^3$, thus it is enough to prove this for C^r maps from $[0,1]$ to $\mathbb{R}^3$. Futhermore, considering each component, our construction reduces to applying the following lemma with $\ell = r+1$ and the construction will be complete.

Lemma 4.2 *Given $\delta > 0$ and $\ell \in \mathbb{N}$ consider the set*

$$\mathcal{W}_{\delta\,,\ell} = \{g \in C^r([0,1],\mathbb{R}) \mid ||g||_r < \delta\,,\ g^{(k)}(0) = 0\,,\ 0 \leq k \leq \ell \text{ and } g^{(k)}(1) = 0\,,\ 1 \leq k \leq \ell\},.$$

Then the image of $\mathcal{W}_{\delta\,,\ell}$ under the evaluation map $g \to g(1) : \mathcal{W}_{\delta\,,\ell} \to \mathbb{R}$ contains a ball of radius $K_2\delta$, (where the constant $K_2 \equiv K_2(\ell)$ depends only on ℓ).

Proof: An explicit construction of such a function can be given along the lines of 'polynomial interpolation'. Consider $g(x) = \sum_{k=\ell+1}^{2\ell+1} a_k x^k$. Clearly g has the desired $(0,\ell)$-jet at 0. We want the $(0,\ell)$-jet of g at 1 to be of the form $(\xi, 0, 0, \cdots, 0)$, for any ξ close enough to 0. The map $(a_{\ell+1}, \cdots, a_{2\ell+1}) \to (g(1), g'(1), \cdots, g^{(\ell)}(1)) : \mathbb{R}^{\ell+1} \to \mathbb{R}^{\ell+1}$ is linear and it is easy to see that the j^{th} column v_j of the matrix $P = (p_{ij})$ representing it is given by (the transpose of), $(1\,, \ell{+}j\,, (\ell{+}j)(\ell{+}j{-}1)\,, \cdots, (\ell{+}j)(\ell{+}j{-}1)\cdots j)$. We claim that this matrix is nonsingular. To show this, suppose $\sum_{j=1}^{\ell+1} \mu_j v_j = 0$ for some scalars μ_j, $(1 \leq j \leq \ell+1)$. This implies that (considering the first entry of this vector), $\sum_{j=1}^{\ell+1} \mu_j = 0$, and (by considering the i^{th} entry of this vector), $\sum_{j=1}^{\ell+1} \mu_j(\ell+j)(\ell+j-1)\cdots(\ell+j-i) = 0$, for $1 \le i \le \ell$. A suitable combinations of these equations show that

$$\sum_{j=1}^{\ell+1} \mu_j \, j^p = 0\,, \quad \text{for } p = 0\,,1\,,2\,,\cdots,\ell\,,$$

This system of equations is just the linear system $V\bar{\mu} = 0$, where $V = (V_{ij})$ is the $(\ell+1)\times(\ell+1)$ Vandermonde matrix given by $v_{ij} = j^i$, $(0 \le i \le \ell$ and $1 \le j \le \ell+1)$ and $\bar{\mu} = (\mu_1, \cdots, \mu_{\ell+1})^t$. It is well known that this Vandermonde matrix V is nonsingular. Hence given $\delta > 0$, the image of the δ ball in $\mathbb{R}^{\ell+1}$ centered at the zero vector contains a ball of radius $\frac{\delta}{||V^{-1}||}$ centered at zero.. Thus, the lemma is proved and K_2^* can be taken to be $\frac{1}{||V^{-1}||}$, which depends only on ℓ. ∎

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