Notes on Schneider's Stability Estimates for Convex Sets in Minkowski Space

Gabor Toth
Department of Mathematics
Rutgers University
Camden, New Jersey 08102
e-mail: gtoth@camden.rutgers.edu

Abstract

In 2009 [Schneider 1] obtained stability estimates in terms of the Banach-Mazur distance for several geometric inequalities for convex bodies in an n-dimensional Minkowski space $\mathbb{E}^n$. A unique feature of his approach is to express fundamental geometric quantities in terms of a single function $\rho : \mathfrak{B} \times \mathfrak{B} \to \mathbb{R}$ defined on the set of all convex bodies $\mathfrak{B}$ in $\mathbb{E}^n$. In this paper we show that (the logarithm of) the symmetrized ρ gives rise to a pseudo-metric d_D on $\mathfrak{B}$ inducing a finer topology than Banach-Mazur's d_{BM}. Further, d_D induces a metric on the quotient $\mathfrak{B}/\mathrm{Dil}^+$ of $\mathfrak{B}$ by the relation of positive dilatation (homothety). Unlike its compact Banach-Mazur counterpart, d_D is only "boundedly compact," in particular, complete and locally compact. The general linear group $\mathrm{GL}(\mathbb{E}^n)$ acts on $\mathfrak{B}/\mathrm{Dil}^+$ by isometries with respect to d_D, and the orbits space is naturally identified with the Banach-Mazur compactum $\mathfrak{B}/\mathrm{Aff}$ via the natural projection $\pi : \mathfrak{B}/\mathrm{Dil}^+ \to \mathfrak{B}/\mathrm{Aff}$, where Aff is the affine group of $\mathbb{E}^n$. The metric d_D has the advantage that many geometric quantities are explicitly computable. We will show that d_D provides a simpler and more fitting environment for the study of stability; in particular, all the estimates of [Schneider 1] turn out to be valid with d_{BM} replaced by d_D.

1 A Positive-Dilatation Invariant Pseudo-Metric

Let $\mathbb{E}^n$ be a Minkowski space of dimension n, and denote by $\mathfrak{B}$ the set of all convex bodies in $\mathbb{E}^n$. We emphasize here that we only consider convex bodies that have non-empty interior in $\mathbb{E}^n$, that is, all the members of $\mathfrak{B}$ have dimension n.

Let $\mathrm{Aff} = \mathrm{Aff}\,(\mathbb{E}^n)$ denote the affine group, the Lie group of affine transformations of $\mathbb{E}^n$. (For brevity $\mathbb{E}^n$ will be suppressed from the notation.) It can be written as the semi-direct product $\mathrm{Aff} = \mathrm{T} \rtimes \mathrm{GL}$, where $\mathrm{T} \cong \mathbb{E}^n$ is the (additive) group of translations of $\mathbb{E}^n$ and GL is the general linear group of $\mathbb{E}^n$. The affine group Aff acts naturally on $\mathfrak{B}$.

The (extended) Banach-Mazur distance function $d_{BM} : \mathfrak{B} \times \mathfrak{B}$ is defined, for $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$, as

$$d_{BM}(\mathcal{C}, \mathcal{C}') = \min\{\alpha \geq 1 \,|\, \mathcal{C} \subset \phi(\mathcal{C}') \subset \alpha\mathcal{C} + Z \text{ for some } \phi \in \text{ Aff and } Z \in \mathbb{E}^n\}. \quad (1)$$

It is an easy exercise to show that d_{BM} satisfies the following properties: (i) $d_{BM}(\mathcal{C}, \mathcal{C}') = 1$ for $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ if and only if $\mathcal{C}' = \phi(\mathcal{C})$ for some $\phi \in$ Aff; (ii) Symmetry: $d_{BM}(\mathcal{C}, \mathcal{C}') = d_{BM}(\mathcal{C}', \mathcal{C})$ for any $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$; (iii) Multiplicativity: $d_{BM}(\mathcal{C}, \mathcal{C}'') \leq d_{BM}(\mathcal{C}, \mathcal{C}') \cdot d_{BM}(\mathcal{C}', \mathcal{C}'')$ for any $\mathcal{C}, \mathcal{C}', \mathcal{C}'' \in \mathfrak{B}$; (iv) Affine-invariance: $d_{BM}(\phi(\mathcal{C}), \phi'(\mathcal{C}')) = d_{BM}(\mathcal{C}, \mathcal{C}')$ for any $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ and $\phi, \phi' \in$ Aff.

It follows that $\ln(d_{BM})$ is a metric on the quotient $\mathfrak{B}/\,\text{Aff}$.

Remark. The extended Banach-Mazur distance function is sometimes called the *Minkowski distance* or *affine distance*. Originally, d_{MB} was defined only for symmetric convex bodies.

There are several deep results in connection with the Banach-Mazur metric properties of $\mathfrak{B}/\,\text{Aff}$. In 1948 in a pioneering work [Fritz John] proved that every convex body $\mathcal{C} \in \mathfrak{B}$ possesses a unique ellipsoid $\mathcal{E} \in \mathfrak{B}$ of maximal volume such that

$$\mathcal{E} \subset \mathcal{C} \subset n(\mathcal{E} - c) + c, \quad (2)$$

where c is the centroid (center) of $\mathcal{E}$. In addition, for $\mathcal{C}$ symmetric, the scaling factor n can be improved to $\sqrt{n}$.

Using the Banach-Mazur distance, we thus have $d_{BM}(\mathcal{C}, \mathcal{E}) \leq n$ or $\leq \sqrt{n}$, for symmetric $\mathcal{C}$. Since any two ellipsoids are affine equivalent, properties (iii)-(iv) imply that $d_{BM}(\mathcal{C}, \mathcal{C}') \leq n^2$ for any $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$, and, in addition, $d_{BM}(\mathcal{C}, \mathcal{C}') \leq n$ provided that $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ are both symmetric. By Blaschke's Selection Theorem, $\mathfrak{B}/\text{Aff}$ is complete, and hence compact. (See [Schneider 2] and also Theorem 1 below.)

There has been extensive work in finding the best possible bounds in John's estimates. See [Rudelson, Lassak, Gluskin], and also the unified approach by [Guo-Kaijser].

An affine transformation ϕ of $\mathbb{E}^n$ is called a dilatation (or homothety) if $\phi(C) = \alpha C + Z$, $C \in \mathbb{E}^n$, for some $\alpha \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $Z \in \mathbb{E}^n$. It is a positive dilatation if $\alpha > 0$. Using this, $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ are related by a (positive) dilatation (or homothetic) if there exists a (positive) dilatation ϕ such that $\mathcal{C}' = \phi(\mathcal{C})$.

The group of dilatations in the affine group is denoted by Dil $\subset$ Aff. Restricting to positive dilatations, we obtain the subgroup $\text{Dil}^+ \subset$ Dil.

Using the semi-direct product $\text{Aff} = \text{T} \rtimes \text{GL}$, if $L : \text{Aff} \to GL$ denotes the natural homomorphism with kernel T, then Dil (resp. Dil^+) is the inverse image of the center

$\mathbb{R}^* \cdot I \subset GL$ (resp. $\mathbb{R}^+ \cdot I$) under L. Clearly, Dil^+ is the identity component of Dil. With respect to the action of Aff on $\mathfrak{B}$, the subgroup Dil^+ acts on $\mathfrak{B}$ freely.

In analogy with the Banach-Mazur metric, for $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$, we define

$$d_D(\mathcal{C}, \mathcal{C}') = \min\{\alpha \geq 1 \,|\, \mathcal{C} \subset \phi(\mathcal{C}') \subset \alpha\mathcal{C} + Z \text{ for some } \phi \in \mathrm{Dil}^+ \text{ and } Z \in \mathbb{E}^n\}. \quad (3)$$

As for the Banach-Mazur metric, it follows that d_D satisfies properties (i)-(iv) with the affine group Aff replaced by Dil^+. In particular, d_D induces a metric on the quotient $\mathfrak{B}/\mathrm{Dil}^+$.

By definition, we also have

$$d_{BM}(\mathcal{C}, \mathcal{C}') = \inf\{d_D(\mathcal{C}, \phi(\mathcal{C}')) \,|\, \phi \in \mathrm{Aff}\} \leq d_D(\mathcal{C}, \mathcal{C}'), \; \mathcal{C}, \mathcal{C}' \in \mathfrak{B}. \quad (4)$$

Indeed, comparing (1) and (3), $\alpha \geq d_{BM}(\mathcal{C}, \mathcal{C}')$ implies $\alpha \geq d_D(\mathcal{C}, \phi(\mathcal{C}'))$, for some $\phi \in \mathrm{Aff}$, so that $d_{BM}(\mathcal{C}, \mathcal{C}')$ is greater than equal to the infimum in (4). The reverse inequality follows from affine invariance of d_{BM}.

Following [Schneider 1], we now introduce the function $\rho : \mathfrak{B} \times \mathfrak{B} \to \mathbb{R}$ by

$$\rho(\mathcal{C}, \mathcal{C}') = \min\{\lambda > 0 \,|\, \mathcal{C} + X \subset \lambda\mathcal{C}' \text{ for some } X \in \mathbb{E}^n\}, \; \mathcal{C}, \mathcal{C}' \in \mathfrak{B}.$$

Our first observation is the following:

Proposition 1. *The function ρ is sub-multiplicative:*

$$\rho(\mathcal{C}, \mathcal{C}'') \leq \rho(\mathcal{C}, \mathcal{C}') \cdot \rho(\mathcal{C}', \mathcal{C}''), \; \mathcal{C}, \mathcal{C}', \mathcal{C}'' \in \mathfrak{B}.$$

PROOF. Let $\lambda \geq \rho(\mathcal{C}, \mathcal{C}')$ and $\lambda' \geq \rho(\mathcal{C}', \mathcal{C}'')$ so that we have

$$\mathcal{C} + X \subset \lambda\mathcal{C}' \quad \text{and} \quad \mathcal{C}' + X' \subset \lambda'\mathcal{C}'', \text{ for some } X, X' \in \mathbb{E}^n. \quad (5)$$

Combining these, we obtain

$$\mathcal{C} + X + \lambda X' \subset \lambda\mathcal{C}' + \lambda X' \subset \lambda\lambda'\mathcal{C}''. \quad (6)$$

Thus, we have $\lambda\lambda' \geq \rho(\mathcal{C}, \mathcal{C}'')$. The proposition follows.

Our second and crucial observation is that d_D is the symmetrized Schneider function ρ:

Proposition 2. *We have*

$$d_D(\mathcal{C},\mathcal{C}') = \rho(\mathcal{C},\mathcal{C}') \cdot \rho(\mathcal{C}',\mathcal{C}),\ \ \mathcal{C},\mathcal{C}' \in \mathfrak{B}. \tag{7}$$

Proof. Let $\mathcal{C},\mathcal{C}' \in \mathfrak{B}$. Assume that $\lambda \geq \rho(\mathcal{C},\mathcal{C}')$ and $\lambda' \geq \rho(\mathcal{C}'\mathcal{C})$ so that(5) holds with $\mathcal{C}'' = \mathcal{C}$. Then (6) gives

$$\mathcal{C} \subset \lambda\mathcal{C}' - X \subset \lambda\lambda'\mathcal{C} - X - \lambda X'.$$

Setting $\alpha = \lambda\lambda'$, $Z = -X - \lambda X'$ and $\phi(C') = \lambda C' - X$, $C' \in \mathbb{E}^n$, we have $\phi \in \mathrm{Dil}^+$, and hence $d_D(\mathcal{C},\mathcal{C}') \leq \alpha = \lambda\lambda'$. We obtain $d_D(\mathcal{C},\mathcal{C}') \leq \rho(\mathcal{C},\mathcal{C}') \cdot \rho(\mathcal{C}',\mathcal{C})$.
For the reverse inequality, assume that in the definition of d_D in (3) we have

$$\mathcal{C} \subset \phi(\mathcal{C}') \subset \alpha\mathcal{C} + Z,$$

for some $\phi \in \mathrm{Dil}^+$, $\alpha \geq 1$ and $Z \in \mathbb{E}^n$. Since ϕ is a positive dilatation, we have $\phi(C') = \lambda C' + X$, $C' \in \mathbb{E}^n$, for some $\lambda > 0$ and $X \in \mathbb{E}^n$. The inclusions above then give $\rho(\mathcal{C},\mathcal{C}') \leq \lambda$ and $\rho(\mathcal{C}',\mathcal{C}) \leq \alpha/\lambda$. Thus, $d_D(\mathcal{C},\mathcal{C}') \leq \alpha$, and we obtain $d_D(\mathcal{C},\mathcal{C}') \geq \rho(\mathcal{C},\mathcal{C}') \cdot \rho(\mathcal{C}',\mathcal{C})$. The proposition follows.

One of the principal advantages of the function ρ is its computability in a number of specific instances. In fact, as [Schneider 1] pointed out, the four basic metric invariants of a convex body $\mathcal{C} \in \mathfrak{B}$: the circumradius $R_{\mathcal{C}}$, the inradius $r_{\mathcal{C}}$, the diameter $D_{\mathcal{C}}$, and the minimal width $d_{\mathcal{C}}$ can be expressed by ρ as follows:

$$R_{\mathcal{C}} = \rho(\mathcal{C},\mathcal{B}), \tag{8}$$

$$r_{\mathcal{C}} = \frac{1}{\rho(\mathcal{B},\mathcal{C})}, \tag{9}$$

$$D_{\mathcal{C}} = 2\rho(\mathcal{C}^*,\mathcal{B}), \tag{10}$$

$$d_{\mathcal{C}} = \frac{2}{\rho(\mathcal{B},\mathcal{C}^*)}, \tag{11}$$

where $\mathcal{B} \subset \mathbb{E}^n$ is the unit ball, and $\mathcal{C}^* = (\mathcal{C} - \mathcal{C})/2$ is the Minkowski symmetral of $\mathcal{C}$.

As an immediate consequence of (7)-(11), for $\mathcal{C} \in \mathfrak{B}$, we have

$$d_D(\mathcal{C},\mathcal{B}) = \rho(\mathcal{C},\mathcal{B})\rho(\mathcal{B},\mathcal{C}) = \frac{R_{\mathcal{C}}}{r_{\mathcal{C}}} \tag{12}$$

$$d_D(\mathcal{C}^*,\mathcal{B}) = \rho(\mathcal{C}^*,\mathcal{B})\rho(\mathcal{B},\mathcal{C}^*) = \frac{D_{\mathcal{C}}}{d_{\mathcal{C}}}. \tag{13}$$

In particular, taking ellipsoids, we see that d_D is unbounded on $\mathfrak{B}/\mathrm{Dil}^+$.

For the next step recall the Hausdorff distance

$$d_H(\mathcal{C},\mathcal{C}') = \inf\{r \geq 0 \,|\, \mathcal{C} \subset \mathcal{C}' + r\bar{\mathcal{B}},\, \mathcal{C}' \subset \mathcal{C} + r\bar{\mathcal{B}}\},\ \mathcal{C},\mathcal{C}' \in \mathfrak{B}.$$

Proposition 3. *Let $\mathcal{C},\mathcal{C}' \in \mathfrak{B}$ with respective inradii $r_{\mathcal{C}}, r_{\mathcal{C}'} \geq 1$. Then we have*

$$d_D(\mathcal{C},\mathcal{C}') \leq (1 + d_H(\mathcal{C},\mathcal{C}'))^2, \tag{14}$$

in particular

$$\ln d_D(\mathcal{C},\mathcal{C}') \leq 2 d_H(\mathcal{C},\mathcal{C}').$$

PROOF. Given $r \geq d_H(\mathcal{C},\mathcal{C}')$, we have

$$\mathcal{C} \subset \mathcal{C}' + r\bar{\mathcal{B}} \quad \text{and} \quad \mathcal{C}' \subset \mathcal{C} + r\bar{\mathcal{B}}.$$

Let $O_{\mathcal{C}}$ and $O_{\mathcal{C}'}$ be the incenters of $\mathcal{C}$ and $\mathcal{C}'$, respectively. By definition and the assumption $r_{\mathcal{C}}, r_{\mathcal{C}'} \geq 1$, we have $\bar{\mathcal{B}} \subset \mathcal{C} - O_{\mathcal{C}}$ and $\bar{\mathcal{B}} \subset \mathcal{C}' - O_{\mathcal{C}'}$. Combining all these, we have

$$\begin{aligned} \mathcal{C} + rO_{\mathcal{C}'} &\subset \mathcal{C}' + r\bar{\mathcal{B}} + rO_{\mathcal{C}'} \subset (1+r)\mathcal{C}', \\ \mathcal{C}' + rO_{\mathcal{C}} &\subset \mathcal{C} + r\bar{\mathcal{B}} + rO_{\mathcal{C}} \subset (1+r)\mathcal{C}. \end{aligned}$$

Hence $d(\mathcal{C},\mathcal{C}') = \rho(\mathcal{C},\mathcal{C}')\rho(\mathcal{C}',\mathcal{C}) \leq (1+r)^2$, and (14) follows.

Remark. A similar argument shows that, for $\mathcal{C},\mathcal{C}' \in \mathfrak{B}$, with respective circumradii $R_{\mathcal{C}}, R_{\mathcal{C}'} \leq 1$, we have

$$1 + d_H^T(\mathcal{C},\mathcal{C}') \leq d_D(\mathcal{C},\mathcal{C}'),$$

where

$$d_H^T(\mathcal{C},\mathcal{C}') = \inf\{d_H(\mathcal{C},\mathcal{C}' + Z) \,|\, Z \in \mathbb{E}^n\}$$

is the translation invariant Hausdorff distance. Note that d_H^T is not dilatation invariant.

Theorem 1. *The metric $\ln(d_D)$ is boundedly compact on the quotient $\mathfrak{B}/\mathrm{Dil}^+$. In particular, it makes $\mathfrak{B}/\mathrm{Dil}^+$ a complete and locally compact metric space.*

PROOF. Let $(\mathcal{C}_k)_{k\geq 1} \subset \mathfrak{B}$ be a d_D-bounded sequence, that is, there exists $R \geq 1$ such that $d_D(\mathcal{C}_k, \mathcal{B}) \leq R$ for $k \geq 1$. Since d_D is invariant under positive dilatations, we may assume that, for $k \geq 1$, the inradius $r_{\mathcal{C}_k} = 1$ and the circumcenter of $\mathcal{C}_k$ is at the origin.

By Proposition 2 and (12), we have $d_D(\mathcal{C}_k, \mathcal{B}) = R_{\mathcal{C}_k}/r_{\mathcal{C}_k} = R_{\mathcal{C}_k} \leq R$. We obtain that the sequence $(\mathcal{C}_k)_{k\geq 1}$ is bounded. By Blaschke's Selection Theorem [Schneider 2], a subsequence $(\mathcal{C}_{k_l})_{l\geq 1}$ converges to a convex body $\mathcal{C} \in \mathfrak{B}$ in the Hausdorff metric d_H. Since $r_{\mathcal{C}_{k_l}} = 1$, we have $r_{\mathcal{C}} = 1$ and Proposition 3 applies. We obtain $d_D(\mathcal{C}_{k_l}, \mathcal{C}) \to 1$ as $l \to \infty$. The theorem follows.

Remark. A similar argument in the use of Proposition 3 and (4) shows that $\mathfrak{B}/\mathrm{Aff}$ is complete (and hence compact) with respect to d_{BM}.

Since $\mathrm{Dil}^+ \subset \mathrm{Aff}$ is a normal subgroup, the quotient $\mathrm{Aff}/\mathrm{Dil}^+$ acts on $\mathfrak{B}/\mathrm{Dil}^+$ naturally. (As Lie groups, we have $\mathrm{Aff}/\mathrm{Dil}^+ \cong \mathrm{GL}/(\mathbb{R}^+ \cdot I)$, a double cover of the projective general linear group $\mathrm{PGL} = \mathrm{GL}/(\mathbb{R}^* \cdot I)$.) By (4), the natural projection $\pi : \mathfrak{B}/\mathrm{Dil}^+ \to \mathfrak{B}/\mathrm{Aff}$ is continuous and open with respect to the metrics $\ln(d_D)$ and $\ln(d_{BM})$. (The projection of a d_D-metric ball in $\mathfrak{B}/\mathrm{Dil}^+$ is a d_{BM}-metric ball of the same radius in $\mathfrak{B}/\mathrm{Aff}$.) Thus, the quotient topology by π is the Banach-Mazur topology.

The general linear group GL acts naturally on $\mathfrak{B}/\mathrm{Dil}^+$. Indeed, assume that $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ are related by a positive dilatation: $\mathcal{C}' = \alpha\mathcal{C} + Z$, $\alpha > 0$, $Z \in \mathbb{E}^N$. Applying any $\psi \in \mathrm{GL}$ to both sides, we obtain $\psi(\mathcal{C}') = \psi(\alpha\mathcal{C} + Z) = \alpha\psi(\mathcal{C}) + \psi(Z)$ and the claim follows.

Moreover, this action of GL on $\mathfrak{B}/\mathrm{Dil}^+$ is by isometries with respect to the metric d_D. Indeed, let $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$, and consider the defining relation $\mathcal{C} \subset \phi(\mathcal{C}') \subset \alpha\mathcal{C} + Z$, $\phi \in \mathrm{Dil}^+$, $\alpha \geq 1$, $Z \in \mathbb{E}^n$, in the definition (3) of $d_D(\mathcal{C}, \mathcal{C}')$. Applying any $\psi \in \mathrm{GL}$, we obtain

$$\psi(\mathcal{C}) \subset \psi(\phi(\mathcal{C}')) = \phi^\psi(\psi(\mathcal{C}')) \subset \psi(\alpha\mathcal{C} + Z) = \alpha\psi(\mathcal{C}) + \psi(Z),$$

where $\phi^\psi = \psi\phi\psi^{-1} \in \mathrm{Dil}^+$ (since Dil^+ is normal in Aff). We obtain that

$$d_D(\psi(\mathcal{C}), \psi(\mathcal{C}')) = d_D(\mathcal{C}, \mathcal{C}'),\ \mathcal{C}, \mathcal{C}' \in \mathfrak{B},\ \psi \in \mathrm{GL},$$

and the claim follows.

Finally, note that the GL-orbits on $\mathfrak{B}/\mathrm{Dil}^+$ are precisely the fibres of the projection $\pi : \mathfrak{B}/\mathrm{Dil}^+ \to \mathfrak{B}/\mathrm{Aff}$. (This is because the afffine group is generated by GL and Dil^+.)

To obtain a finer structure on $\mathfrak{B}/\mathrm{Dil}^+$, we introduce an equivalence relation $\sim$ on $\mathfrak{B}$ as follows: $\mathcal{C} \sim \mathcal{C}'$, $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$, if, up to a positive dilatation, $\mathcal{C}$ and $\mathcal{C}'$ have the same John ellipsoid. Clearly, $\sim$ depends only on the positive dilatation classes of $\mathcal{C}$ and $\mathcal{C}'$ so that it induces an equivalence relation $\sim$ on the quotient $\mathfrak{B}/\mathrm{Dil}^+$. Each equivalence class is represented by an ellipsoid $\mathcal{E}$ shared (up to positive dilatation) by every member of the class as its John ellipsoid. We make $\mathcal{E}$ unique by setting its center at the origin and its inradius one.

By the definition of the John ellipsoid, the equivalence classes are d_D-closed. In addition, if $\mathcal{E}$ is the John ellipsoid of $\mathcal{C} \in \mathfrak{B}$ then, by (2), $d_D(\mathcal{C}, \mathcal{E}) \leq n$. We obtain that the diameters of the equivalence classes are uniformly d_D-bounded by $2\ln(n)$. By Theorem 1 above, it follows that the equivalence classes are compact.

Since all ellipsoids are affine equivalent, π maps every equivalence class onto $\mathfrak{B}/\mathrm{Aff}$.

Now let $\mathfrak{C} \subset \mathfrak{B}/\mathrm{Dil}^+$ be an equivalence class with a common John ellipsoid $\mathcal{E}$. Let $\mathrm{Aff}_{\mathcal{E}} \subset \mathrm{Aff}$ be the stabilizer of $\mathcal{E} \in \mathfrak{B}$ (with respect to the action of Aff on $\mathfrak{B}$). By definition, for $\psi \in \mathrm{Aff}_{\mathcal{E}}$, we have $\psi(\mathcal{E}) = \mathcal{E}$; in particular, ψ (acting on $\mathbb{E}^n$) fixes the origin, the centroid of $\mathcal{E}$. We obtain that $\mathrm{Aff}_{\mathcal{E}} \subset \mathrm{GL}$. By the discussion above $\mathrm{Aff}_{\mathcal{E}}$ acts on $\mathfrak{C}$ by isometries (with respect to d_D.)

In addition, $\mathrm{Aff}_{\mathcal{E}}$ (acting on $\mathbb{E}^n$) is transitive on $\mathcal{E}$. Thus, its action on $\mathfrak{C}$ has the unique fixed point $\mathcal{E} \in \mathfrak{B}$. Every other $\mathrm{Aff}_{\mathcal{E}}$-orbit in $\mathfrak{C}$ is at a constant d_D-distance from $\mathcal{E}$.

We finally claim that the orbits of the action of $\mathrm{Aff}_{\mathcal{E}}$ on $\mathfrak{C}$ are precisely the fibres of the restriction $\pi \,|\, \mathfrak{C}$. Indeed, if the Dil^+-orbits of two convex bodies $\mathcal{C}'$ and $\mathcal{C}''$ with common John ellipsoid $\mathcal{E}$ are mapped to to the same affine orbit by π, then there is an affine transformation $\phi \in \mathrm{Aff}$ such that $\phi(\mathcal{C}') = \mathcal{C}''$. By unicity of the John ellipsoid, we have $\phi(\mathcal{E}) = \mathcal{E}$ so that $\phi \in \mathrm{Aff}_{\mathcal{E}}$. The converse is clear.

We summarize the structure of $\mathfrak{B}/\mathrm{Dil}^+$ in the following:

Theorem 2. *(i) The general linear group* GL *acts naturally on* $\mathfrak{B}/\mathrm{Dil}^+$ *by isometries with respect to the metric* d_D*, and the orbits are the fibres of the continuous and open projection* π *of* $\mathfrak{B}/\mathrm{Dil}^+$ *to the Banach-Mazur compactum* $\mathfrak{B}/\mathrm{Aff}$.
(ii) $\mathfrak{B}/\mathrm{Dil}^+$ *is partitioned into compact equivalence classes induced by the relation on the convex bodies having the same John ellipsoid (up to positive dilatation). Given an equivalence class* $\mathfrak{C} \subset \mathfrak{B}/\mathrm{Dil}^+$ *with John ellipsoid* $\mathcal{E}$ *(centered at the origin and having inradius one), the stabilizer* $\mathrm{Aff}_{\mathcal{E}} \subset \mathrm{GL}$ *acts on* $\mathfrak{C}$ *with the unique fixed point* $\mathcal{E} \in \mathfrak{B}$ *and all other orbits are contained in concentric* d_D*-spheres with center at* $\mathcal{E}$*. The orbit space of this action is the Banach-Mazur compactum* $\mathfrak{B}/\mathrm{Aff}$*, and the orbit map is the restriction of the natural projection* $\pi : \mathfrak{B}/\mathrm{Dil}^+ \to \mathfrak{B}/\mathrm{Aff}$ *to* $\mathfrak{C}$.

2 The Minkowski Measure and Schneider's ρ Function

Minkowski's measure of (a)symmetry is defined by

$$\mu(\mathcal{C}) = \min\{\lambda > 0 \,|\, \mathcal{C} + X \subset -\lambda\mathcal{C} \text{ for some } X \in \mathbb{E}^n\}, \ \mathcal{C} \in \mathfrak{B}.$$

(See [Grünbaum, Schneider 1, Schneider 2].) By the definition of ρ, we immediately have

$$\mu(\mathcal{C}) = \rho(\mathcal{C}, -\mathcal{C}), \ \mathcal{C} \in \mathfrak{B}. \tag{15}$$

We will make use of the classical Minkowski-Radon inequality

$$1 \leq \mu \leq n. \tag{16}$$

The lower bound is attained, $\mu(\mathcal{C}) = 1$, if and only if $\mathcal{C}$ is symmetric. The upper bound is attained, $\mu(\mathcal{C}) = n$, if and only if $\mathcal{C}$ is a simplex. (The statement on the lower bound is clear. For a list of references of classical proofs for the upper estimate, see [Grünbaum].)

In addition, following [Schneider 1] again, forming differences of $\mathcal{C} \in \mathfrak{B}$ in the defining inequality of ρ in two ways (to obtain the Minkowski symmetral $\mathcal{C}^*$) gives

$$\rho(\mathcal{C}, \mathcal{C}^*) = \frac{2\mu(\mathcal{C})}{\mu(\mathcal{C}) + 1} \tag{17}$$

$$\rho(\mathcal{C}^*, \mathcal{C}) = \frac{\mu(\mathcal{C}) + 1}{2}. \tag{18}$$

Proposition 4. *For $\mathcal{C} \in \mathfrak{B}$, we have*

$$\frac{R_{\mathcal{C}}}{D_{\mathcal{C}}} \leq \frac{n}{n+1} \quad \text{and} \quad \frac{d_{\mathcal{C}}}{r_{\mathcal{C}}} \leq n+1. \tag{19}$$

Proof. Using (8)-(11) and (17)-(18) along with sub-multiplicativity of ρ (Proposition 1), we have

$$\frac{R_{\mathcal{C}}}{D_{\mathcal{C}}} = \frac{\rho(\mathcal{C}, \mathcal{B})}{2\rho(\mathcal{C}^*, \mathcal{B})} \leq \frac{\rho(\mathcal{C}, \mathcal{C}^*)}{2} = \frac{\mu(\mathcal{C})}{\mu(\mathcal{C}) + 1} \leq \frac{n}{n+1} \tag{20}$$

$$\frac{d_{\mathcal{C}}}{r_{\mathcal{C}}} = \frac{2\rho(\mathcal{B}, \mathcal{C})}{\rho(\mathcal{B}, \mathcal{C}^*)} \leq 2\rho(\mathcal{C}^*, \mathcal{C}) = \mu(\mathcal{C}) + 1 \leq n+1. \tag{21}$$

In the last inequalities we used monotonicity and the Minkowski-Radon inequality (16).

Remark. The first inequality in (19) is due to [Bohnenblust] in 1938. Both estimates in (19) have been proved by [Leichtweiss] in 1955, and a few years later independent proofs have been given by [Eggleston]. In Euclidean space there are sharper estimates due to [Jung] and [Steinhagen], respectively. In Minkowski space the upper bounds in (19) are sharp and attained on any simplex Δ whose difference body $\Delta - \Delta = \mathcal{B} \subset \mathbb{E}^n$ is the unit ball. Conversely, if equality holds for a convex body $\mathcal{C} \in \mathfrak{B}$ in either of the inequalities in (19) then $\mathcal{C}$ is still a simplex with some specific properties as described by [Leichtweiss, Satz 2 and Satz 3].

Finally, we note the universal estimate in [Schneider 1] on pairs of convex bodies:

$$\frac{\rho(\mathcal{C}, \mathcal{C}')}{\rho(\mathcal{C}^*, \mathcal{C}'^*)} \leq n, \ \mathcal{C}, \mathcal{C}' \in \mathfrak{B} \tag{22}$$

This is also a consequence of sub-multiplicativity of ρ (applied twice) as

$$\frac{\rho(\mathcal{C}, \mathcal{C}')}{\rho(\mathcal{C}^*, \mathcal{C}'^*)} \leq \rho(\mathcal{C}, \mathcal{C}^*)\rho(\mathcal{C}'^*, \mathcal{C}') \leq \frac{2\mu(\mathcal{C})}{\mu(\mathcal{C})+1}\frac{\mu(\mathcal{C}')+1}{2} \leq \frac{n}{n+1}(n+1) \leq n. \tag{23}$$

If the upper bound is attained then we immediately see that $\mu(\mathcal{C}) = \mu(\mathcal{C}') = n$ so that $\mathcal{C}$ and $\mathcal{C}'$ are both simplices. In addition, as shown by [Schneider 1], $\mathcal{C}'$ must be homothetic to $-\mathcal{C}$, in fact, this characterizes the upper bound n.

3 Stability

Generally speaking, given a universal geometric inequality for convex bodies with extremal values attained by a geometrically well-characterized class of extremal convex bodies, a stability estimate for this inequality quantifies the deviation of a near-extremal convex body from the extremal ones.

The deviation depends on the metric used for the convex bodies. One of the primary aims of these notes is to show that, for many geometric estimates, the metric d_D on $\mathfrak{B}/\mathrm{Dil}^+$ is a better fit than the traditional Banach-Mazur metric.

The simplest (and unfortunately non-illustrative) example is furnished by the lower bound $\mu = 1$ in the Minkowski-Radon inequality (16). This lower bound is attained by symmetric convex bodies. By (17)-(18), we have

$$d_D(\mathcal{C}, \mathcal{C}^*) = \rho(\mathcal{C}, \mathcal{C}^*)\rho(\mathcal{C}^*, \mathcal{C}) = \mu(\mathcal{C}).$$

Since the Minowski symmetral is symmetric, the stability estimate here is a tautology: If the Minkowski measure of $\mathcal{C} \in \mathfrak{B}$ is close to 1 then so is the d_D-distance of $\mathcal{C}$ from its Minkowski symmetral $\mathcal{C}^*$. (Recall that $\ln(d_D)$ is the metric on $\mathfrak{B}/\mathrm{Dil}^+$.) If d_D is replaced by Banach-Mazur's d_{BM} then, by (4) and the above, we have $d_{BM}(\mathcal{C}, \mathcal{C}^*) \leq \mu(\mathcal{C})$, $\mathcal{C} \in \mathfrak{B}$. Once again, a trivial stability estimate follows.

For a genuine and illustrative example, consider the upper estimate in (16). The upper bound $\mu = n$ is attained by simplices. Now the analysis of [Schneider 1, Theorem 2.1] applied to our distance d_D (verbatim) gives the following:

Given $0 \leq \epsilon < 1/n$, we have

$$\mathcal{C} \in \mathfrak{B}: \quad n - \epsilon < \mu(\mathcal{C}) \quad \Rightarrow \quad d_D(\mathcal{C}, \Delta) < 1 + \frac{(n+1)\epsilon}{1 - n\epsilon}, \tag{24}$$

with a suitable simplex $\Delta \in \mathfrak{B}$ (constructed using [Yaglom-Boltyanskiĭ]'s approach to Helly's Theorem). (For previous estimates, see [Guo] and [Böröczky 1], [Böröczky 2].) Then, Schneider's original estimate in d_{BM} follows by (4).

Stability estimates for the classical inequalities in (19) follow directly from (24) using monotonicity of the upper bounds in (20)-(21) in the variable $\mu(C)$:

If a convex body $\mathcal{C} \in \mathfrak{B}$ satisfies one of the conditions

$$\frac{R_{\mathcal{C}}}{D_{\mathcal{C}}} > \frac{n - \epsilon}{n - \epsilon + 1} \quad \text{or} \quad \frac{d_{\mathcal{C}}}{r_{\mathcal{C}}} > n - \epsilon + 1,$$

then, for the same simplex $\Delta \in \mathfrak{B}$ as in (24), we have

$$d_D(\mathcal{C}, \Delta) < 1 + \frac{n+1}{1 - n\epsilon}\epsilon.$$

For a stability estimate of the inequality in (22), assume

$$n - \frac{n}{n+1}\epsilon \leq \frac{\rho(\mathcal{C}, \mathcal{C}')}{\rho(\mathcal{C}^*, \mathcal{C}'^*)}.$$

Then (23) along with $\mu(\mathcal{C}), \mu(\mathcal{C}') \leq n$ imply $\mu(\mathcal{C}) \geq n - n\epsilon$ and $\mu(\mathcal{C}') \geq n - \epsilon$. Using (24) again, we obtain simplices $\Delta, \Delta' \in \mathfrak{B}$ such that

$$d_D(\mathcal{C}, \Delta) \leq 1 + \frac{(n+1)n}{1 - n^2\epsilon}\epsilon \quad \text{and} \quad d_D(\mathcal{C}', \Delta') \leq 1 + \frac{n+1}{1 - n\epsilon}\epsilon.$$

A delicate analysis in [Schneider 1] gives much more: For $0 \leq \epsilon < 1/n(5n^2+1)$, there exists a simplex Δ_0 with centroid at the origin such that

$$d_D(\Delta_0, -\mathcal{C}) \leq \frac{1}{1 - n(5n^2+1)\epsilon} \quad \text{and} \quad d_D(\Delta_0, \mathcal{C}') \leq \frac{1}{1 - 2n\epsilon}.$$

Indeed, his proof gives homothetic copies $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}'$ of $\mathcal{C}$ and $\mathcal{C}'$, respectively, such that

$$(1 - n(5n^2+1)\epsilon)\Delta_0 \subset -\tilde{\mathcal{C}} \subset \Delta_0 \quad \text{and} \quad (1 - 2n\epsilon)\Delta_0 \subset -\tilde{\mathcal{C}}' \subset \Delta_0.$$

Finally, we discuss stability estimates for a sequence of mean Minkowski measures $\{\sigma_{\mathcal{C},m}\}_{m\geq 1}$, $\mathcal{C}\in\mathfrak{B}$, introduced in [Toth 1, 2].

Let $\mathcal{C}\in\mathfrak{B}$ and $O\in\operatorname{int}\mathcal{C}$. For $C\in\partial\mathcal{C}$, denote $\Lambda(C,O)$ the ratio into which O divides the chord in $\mathcal{C}$ passing through C and O with other end-point $C^o\in\partial\mathcal{C}$:

$$\Lambda(C,O)=\frac{d(C,O)}{d(C^o,O)}, \tag{25}$$

where d is the distance on $\mathbb{E}^n$. This defines the distortion function $\Lambda=\Lambda_{\mathcal{C}}:\partial\mathcal{C}\times\operatorname{int}\mathcal{C}\to\mathbb{R}$. (The depencence on $\mathcal{C}$ will be indicated by subscript if necessary.) Clearly, $(C^o)^o=C$ and $\Lambda(C^o,O)=1/\Lambda(C,O)$, $C\in\partial\mathcal{C}$. It follows by an elementary argument that $\Lambda:\partial\mathcal{C}\times\operatorname{int}\mathcal{C}\to\mathbb{R}$ is continuous [Toth 2].

It is well-known that the Minkowski measure of (a)symmetry μ can be expressed via the distortion by

$$\mu(\mathcal{C})=\inf_{O\in\operatorname{int}\mathcal{C}}\max_{C\in\partial\mathcal{C}}\Lambda_{\mathcal{C}}(C,O),\ \mathcal{C}\in\mathfrak{B}. \tag{26}$$

(See, for example [Grünbaum].)

Now, given $m\geq 1$, a multi-set $\{C_0,\ldots,C_m\}\subset\partial\mathcal{C}$ (repetition allowed) is called an m-configuration of $\mathcal{C}$ (with respect to O) if O is contained in the convex hull $\operatorname{conv}(C_0,\ldots,C_m)$. The set of m-configurations of $\mathcal{C}$ is denoted by $\mathfrak{C}_m(O)=\mathfrak{C}_{\mathcal{C},m}(O)$.

We define the function $\sigma_m=\sigma_{\mathcal{C},m}:\operatorname{int}\mathcal{C}\to\mathbb{R}$, as follows

$$\sigma_m(O)=\inf_{\{C_0,\ldots,C_m\}\in\mathfrak{C}_m(O)}\sum_{i=0}^m\frac{1}{\Lambda(C_i,O)+1},\ O\in\operatorname{int}\mathcal{C}. \tag{27}$$

Since Λ is continuous and $\partial\mathcal{C}$ is compact, the infimum is attained. An m-configuration at which $\sigma_m(O)$ attains its minimum is called minimal.

We define

$$\sigma_m^*=\sup_{O\in\operatorname{int}\mathcal{C}}\sigma_m(O),\ m\geq 1.$$

An elementary argument shows that the supremum is attained [Toth 2]. (Clearly, $\sigma_1=\sigma_1^*=1$ identically on $\operatorname{int}\mathcal{C}$.)

Any m-configuration (with respect to an interior point O) can be extended to an $(m+k)$-configuration, $k\geq 1$, by adding k copies of a boundary point of $\mathcal{C}$ at which $\Lambda(.,O)$ attains its maximum. Thus we have the following sub-arithmeticity:

$$\sigma_{m+k}(O)\leq\sigma_m(O)+\frac{k}{\max_{\partial\mathcal{C}}\Lambda(\cdot,O)+1},\ O\in\operatorname{int}\mathcal{C},\ m,k\geq 1. \tag{28}$$

As a direct consequence of Carathéodory's theorem, equality holds for $m = n$ and $k \geq 1$, that is, the sequence $\{\sigma_m\}_{m\geq 1}$ is arithmetic with difference $1/(\max_{\partial\mathcal{C}} \Lambda + 1)$ from the n-th term onwards [Toth 1]. In particular, by (26), we have

$$\lim_{m\to\infty} \frac{\sigma_m^*}{m+1} = \frac{1}{\mu+1}.$$

Finally, we recall the fundamental estimate for the sequence $\{\sigma_m\}_{m\geq 1}$:

$$1 \leq \sigma_m \leq \frac{m+1}{2}, \; m \geq 1. \tag{29}$$

Assuming $m \geq 2$, $\sigma_m(O) = (m+1)/2$ for some $O \in \operatorname{int}\mathcal{C}$ if and only if $\mathcal{C}$ is symmetric with respect to O. If, for some $m \geq 1$, $\sigma_m(O) = 1$ at $O \in \operatorname{int}\mathcal{C}$, then $m \leq n$ and $\mathcal{C}$ has an m-dimensional simplicial intersection across O, that is, there exists an m-dimensional affine subspace $\mathcal{E} \subset \mathbb{E}^n$, $O \in \mathcal{E}$, such that $\mathcal{C} \cap \mathcal{E}$ is an m-simplex. Conversely, if $\mathcal{C}$ has a simplicial intersection with an m-dimensional affine subspace $\mathcal{E}$ then $\sigma = 1$ identically on $\operatorname{int}\mathcal{C} \cap \mathcal{E}$. (For details, see [Toth 2].)

We first derive a stability estimate for the upper bound in (29) as it is much simpler.

Theorem 3. *Let* $2 \leq m \leq n$ *and*

$$0 \leq \epsilon \leq \frac{n-1}{n+1}\frac{m-1}{2}.$$

If $\mathcal{C} \in \mathfrak{B}$ *satisfies*

$$\frac{m+1}{2} - \epsilon \leq \sigma_{\mathcal{C},m}^* \tag{30}$$

then we have

$$d_D(\mathcal{C}, \mathcal{C}^*) \leq 1 + 2\frac{n+1}{m-1}\epsilon. \tag{31}$$

Proof. Sub-arithmeticity in (28) (with $m = 1$ and $k = m - 1$) along with (26) gives

$$\sigma_{\mathcal{C},m}^* \leq 1 + \frac{m-1}{\mu(\mathcal{C})+1}.$$

Combining this with the imposed lower bound (30), we obtain

$$\mu(\mathcal{C}) \leq \frac{2}{1-\delta} - 1 = 1 + 2\frac{\delta}{1-\delta}, \tag{32}$$

where $\delta = 2\epsilon/(m-1)$. The imposed restriction on ϵ translates into

$$0 \leq \delta \leq \frac{n-1}{n+1}.$$

Thus, in (32), we have $2/(1-\delta) \leq n+1$. Finally, as noted above $d_D(\mathcal{C}, \mathcal{C}^*) = \mu(\mathcal{C})$. Putting these back in (32) we obtain (31).

Turning to a stability estimate for the lower bound in (29), recall that $\sigma_{\mathcal{C},m}(O) = 1$ if and only if $\mathcal{C}$ has an m-dimensional simplicial slice across O. Hence a stability estimate can only be expected for $\sigma_{\mathcal{C}} = \sigma_{\mathcal{C},n}$.

As a first attempt, let $0 \leq \epsilon < 1/n(n+1)$ and $\mathcal{C} \in \mathfrak{B}$. We claim that if

$$\sigma^*_{\mathcal{C}} \leq 1 + \epsilon. \tag{33}$$

then there exists a simplex $\Delta \subset \mathcal{C}$ such that

$$d_D(\mathcal{C}, \Delta) < 1 + \frac{(n+1)^2\epsilon}{1 - n\epsilon}. \tag{34}$$

To show this, let $O^* \in \operatorname{int} \mathcal{C}$ be a point at which the infimum in (26) is attained: $\mu(\mathcal{C}) = \max_{\partial\mathcal{C}} \Lambda(\cdot, O^*)$. (The minimal level-set of $\max_{\partial\mathcal{C}} \Lambda$ comprised of these points is the critical set of $\mathcal{C}$, a compact convex set of codimension ≥ 2; see [Klee].) Using the trivial lower bound in (27) for σ, we obtain

$$\frac{n+1}{\mu(\mathcal{C}) + 1} = \frac{n+1}{\max_{\partial\mathcal{C}} \Lambda(\cdot, O^*) + 1} \leq \sigma_{\mathcal{C}}(O^*) \leq \sigma^*_{\mathcal{C}} \leq 1 + \epsilon.$$

Rearranging and estimating, we find

$$n - (n+1)\epsilon \leq \frac{n - \epsilon}{1 + \epsilon} \leq \mu(\mathcal{C}).$$

Now (24) applies (with ϵ replaced by $(n+1)\epsilon$) and (34) follows.

To obtain a stronger stability estimate one needs to relax the inequality in (33).

Theorem 4. *Let $\mathcal{C} \in \mathfrak{B}$ and $O \in \operatorname{int} \mathcal{C}$ satisfying*

$$\max_{\partial\mathcal{C}} \Lambda_{\mathcal{C}}(\cdot, O) \leq n. \tag{35}$$

Assume that, for $0 \leq \epsilon < 1/(n+1)$, we have

$$1 \leq \sigma_{\mathcal{C}}(O) \leq 1 + \epsilon. \tag{36}$$

Then for the convex hull Δ of any minimal configuration we have

$$d_D(\mathcal{C}, \Delta) \leq \frac{1}{1-(n+1)\epsilon}. \tag{37}$$

Proof. We first note that we can lower the value of ϵ (to $\sigma_{\mathcal{C}}(O)-1$), and instead of (36), impose

$$1 < \sigma_{\mathcal{C}}(O) < 1 + \frac{1}{n+1}. \tag{38}$$

(For simplicity, we excluded the trivial case $\sigma_{\mathcal{C}}(O) = 1$.) Assuming now (35) and (38), using a complex construction in [Toth 3, Theorem 1] we showed that, for the convex hull Δ of any minimal configuration, we have

$$\Delta \subset \mathcal{C} \subset \tilde{\Delta} = \tilde{r}(\Delta - \tilde{C}) + \tilde{C}, \tag{39}$$

where

$$\tilde{C} = \frac{1}{\sigma_{\mathcal{C}}(O)-1} \sum_{i=0}^{n} \left(\frac{1}{\Lambda_{\mathcal{C}}(C_i, O)+1} - \frac{1}{\Lambda_{\Delta}(C_i, O)+1} \right) C_i \in \Delta \tag{40}$$

and

$$\tilde{r} = \frac{1}{1-(n+1)(\sigma_{\mathcal{C}}(O)-1)}. \tag{41}$$

Thus, $d_D(\mathcal{C}, \Delta) \leq \tilde{r}$, and we obtain (37) (for $\epsilon = \sigma_{\mathcal{C}}(O)-1$). The theorem follows.

Remarks. 1. According to a classical result of Minkowski, $\Lambda(C, g(\mathcal{C})) \leq n$, $C \in \partial\mathcal{C}$, where $g(\mathcal{C})$ is the centroid of $\mathcal{C}$ [Bonnesen-Fenchel]. Hence (37) holds if $1 < \sigma(g(\mathcal{C})) < 1 + 1/(n+1)$.
2. As the example of the semi-disk shows, the center of similarity $\tilde{C}$ can be on the boundary of $\mathcal{C}$.

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